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ON THE TRANSFORMATION OF THE LINEARIZED EQUATION OF UNSTEADY SUPERSONIC FLOW*

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Summary. The linearized potential equation for unsteady motion in frictionless, supersonic flow is transformed from the classical wave equation to the canonical form $\phi_{xx} - \phi_{yy} - \phi_{zz} = \phi_{\tau\tau}$ with the aid of a modified Lorentz transformation. Possible invariant transformations of the latter, including the classical Lorentz transformation, are discussed. Eleven coordinate systems (each of which has its counterpart in the classical theory of the wave equation) permitting separation of variables are set forth, their derivation being based on the analogy between the hyperbolic metric defined by $(ds)^2 = (dx)^2 - (dy)^2 - (dz)^2$ and the Euclidean (Cartesian) metric. A few practical applications are indicated.

1. Introduction. We consider here the linearized equation for the velocity potential in unsteady, supersonic flow and various coordinate transformations to which it may be usefully subjected in order to effect separation of variables.

In connection with the linearization of the original equations we remark that the assumptions implicit therein impose much stronger restrictions than in such classical fields as electricity and magnetism. Let δ be the fineness ratio, defined as the larger of the maximum thickness of a wing, or the amplitude of transverse motion, divided by the maximum wing chord l, the latter serving as the characteristic length throughout the following analysis. Further let ν be a dimensionless measure of time rate of change, where the characteristic time is lc^{-1}, c being the sonic velocity in the undisturbed medium, and let sl be the average wing span. Then an extension of the two dimensional analysis of Lin, Reissner and Tsien[1] shows that *sufficient* conditions for linearization ($M > 1$) are

$$M\delta \ll 1, \qquad \nu M\delta \ll 1 \tag{1.1}$$

and any *one or more* of

$$|M - 1|, \nu, \quad \text{or} \quad s^{-2} \gg \delta^{2/3}, \tag{1.2}$$

where M is the free stream Mach number.

In the case of slender bodies slightly different restrictions obtain.[2]

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*Received April 1, 1953. The material in this paper is drawn from lectures given while the author was on sabbatical leave in London, particularly at the Imperial College of Science and Technology (February and March, 1952).

[1]C. C. Lin, E. Reissner and H. S. Tsien, J. Math. & Ph. **27**, 220 (1948).

[2]J. W. Miles, J. Aero. Sci. **19**, 380 (1952).

Because of these restrictions the results obtained herein probably have but limited practical application. Nevertheless, we feel that they are of interest *per se* and, in addition, may lead to useful results in the hands of other investigators. Moreover, they may be of some value in attacking the non-linear equations.

2. The potential equation. The linearized potential equation governing small disturbances with respect to a fixed coordinate system in a perfect fluid is (the wave equation)

$$\phi_{XX} + \phi_{YY} + \phi_{ZZ} = \phi_{TT}, \tag{2.1}$$

where X, Y, Z are dimensionless Cartesian coordinates in a *fixed* reference frame, T is a dimensionless time obtained by multiplying the true time (t) by the sonic velocity (c) and dividing by l, and ϕ is the dimensionless velocity potential (reference quantity: Ul). In the case of a supersonic flight at Mach number M along the negative X axis, the body in question may be brought to rest and (1) reduced to (what may be regarded as) a canonical form by introducing the modified Lorentz transformation

$$\begin{pmatrix} x \\ \tau \end{pmatrix} = (M^2 - 1)^{-1/2}\begin{pmatrix} 1 & M \\ M & 1 \end{pmatrix}\begin{pmatrix} X \\ T \end{pmatrix}, \qquad y = Y, \qquad z = Z \tag{2.2}$$

in which case we have

$$\Box\phi = \phi_{xx} - \phi_{yy} - \phi_{zz} = \phi_{\tau\tau}, \tag{2.3}$$

where $\Box$ may be designated as the "hyperbolic Laplacian" operator.*

Equation (3) may be identified as the wave equation in the coordinates (x, iy, iz, τ), just as Bateman[3] has identified the classical wave equation as Laplace's equation, albeit four dimensional, in (X, Y, Z, iT). Such an identification suggests, by analogy, numerous solutions to (3), which, to be sure, must be appropriately restricted if they are to correspond to physical reality.

If in (3) we pose the harmonic dependence† $\exp(-i\kappa\tau)$, with the generality implied by Fourier's theorem, we obtain

$$\Box\phi + \kappa^2\phi = 0 \tag{2.4}$$

which may be appropriately designated as the "hyperbolic Helmholtz equation".

The origin of the moving (x, y, z) coordinates is conveniently chosen at the most upstream point of the body creating the disturbance. Then, in consequence of the supersonic flight velocity, we may assert

$$\phi \equiv 0, \qquad x < 0. \tag{2.5}$$

Accordingly, it is often expedient to introduce the Laplace transformation

$$\Phi = \mathcal{L}\{\phi\} = \int_0^\infty e^{-sx}\phi\, dx \tag{2.6}$$

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*This notation has been used by P. A. Lagerstrom in unpublished lectures at the California Institute of Technology.

[3]H. Bateman, *Partial differential equations*, Dover Publ., New York, 1944, p. 384.

†The choice $\kappa = (M^2 - 1)^{-1/2}(\omega/c)$ yields the dependence $\exp(i\omega t)$ in the original time domain; *cf.* (2.9) *infra.*

which, applied to (4), yields

$$\Phi_{yy} + \Phi_{zz} - \lambda^2\Phi = 0, \tag{2.7}$$

$$\lambda^2 = s^2 + \kappa^2. \tag{2.8}$$

In the application of the end results, it is most convenient to deal with a coordinate x^* measured from the foremost point on the airfoil and the true time t, the corresponding potential being given by

$$\begin{aligned}\phi^*(x^*, y, z, t) &= \phi(x, y, z, \tau)\\ &= \phi[(M^2 - 1)^{-1/2}x^*, y, z, M(M^2 - 1)^{-1/2}x^* - (M^2 - 1)^{1/2}(ct/l)]. \end{aligned}\tag{2.9}$$

However, in all of the subsequent discussion we shall deal implicitly with x and τ.

3. Invariant transformations. The general question of transformations under which the classical wave equation remains invariant has been discussed in a series of papers by Bateman.[4] The corresponding transformations of (2.3) and (2.4) follow *via* the analogy suggested above.

It is immediately evident that (2.3) is invariant under independent translations of (x, y, z, τ), a spherical rotation of (y, z, τ) with x fixed in direction (*N.B.*: (2.3) is not invariant under a rotation involving x, as, *e.g.*, in the case of a transformation to Mach coordinates.), a (simultaneous) scale transformation of (x, y, z, τ), and a scale transformation of ϕ.

Rather less obvious are the inversions studied by Bateman. The simplest of these imply that if $\phi(x, y, z, \tau)$ is a solution to (2.3) so also are

$$\psi(x, y, z, \tau) = \mu\phi(\mu x, \mu y, \mu z, \mu\tau), \tag{3.1}$$

$$\mu = (x^2 - y^2 - z^2 - \tau^2)^{-1}, \tag{3.2}$$

$$\chi(x, y, z, \tau) = \nu\phi[\tfrac{1}{2}\nu(y^2 + z^2 + \tau^2 + 1), \tfrac{1}{2}\nu(y^2 + z^2 + \tau^2 - 1), \nu z, \nu\tau], \tag{3.3}$$

$$\nu = (x - y)^{-1}. \tag{3.4}$$

As an example of a more general result, a homogeneous solution of degree -1 is given by

$$\begin{aligned}\phi(x, y, z, \tau) &= (x + y)^{-\alpha}(x - y)^{-\alpha'}(z + i\tau)^{-\beta}(z - i\tau)^{-\beta'}\\ &\quad (x^2 - y^2 - z^2 - \tau^2)^{-\gamma}(-1)^{-\gamma'}P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}\end{aligned}\tag{3.5}$$

from which additional transformations may be obtained *via* the many transformations of the generalized hypergeometric function P (in the Riemann-Papperitz notation).

Perhaps the most interesting (but not necessarily the most important, since many valuable inferences are afforded by the various scale transformations) transformation under which (2.3) remains invariant is that of Lorentz, which is most conveniently written in the normalized form (so that all transformations obtained by assigning

[4]H. Bateman, Proc. London Math. Soc. (2) **8**, 223 (1909); *ibid* **7**, 70 (1909); **8**, 469 (1910); **10**, 7 (1911).

different values to m are members of a group; *cf.* the work of Lagerstrom[5] and Hayes[6] in steady flow).

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (1 - m^2)^{-1/2} \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad | m | < 1. \tag{3.6}$$

This result has its most direct application in extending a solution for a wing leaving the wing tip $y = 0$ to one making an angle $\tan^{-1} m$ with the direction of flight. (But if m is negative the original tip becomes a trailing edge, and the Kutta condition then must be introduced.) This result has been applied to the problem of a rated wing tip in unsteady flow, starting from the known solution for a rectangular wing and applying the Kutta condition where the edge is trailing (see J. W. Miles, Q. Appl. Math., **11**, 363; 1953).

Additional transformations under which (2.3) remains invariant may be obtained from symmetry considerations among (y, z, τ). Finally, (2.4) is invariant under all of the foregoing transformations not involving τ and to a simultaneous scale transformation of (x, y, z, κ^{-1}).

4. Coordinate transformations. We shall consider only those coordinate transformations

$$x, y, z = x, y, z(q_1, q_2, q_3) \tag{4.1}$$

for which the hyperbolic line element transforms according to

$$(ds)^2 = (dx)^2 - (dy)^2 - (dz)^2 = h_1^2(dq_1)^2 - h_2^2(dq_2)^2 - h_3^2(dq_3)^2 \tag{4.2}$$

where $h_{1,2,3,}$ are positive, real coefficients.* In the sense that the metric is diagonal, these transformations may be said to be orthogonal, but the absence of cross products like q_iq_j does not necessarily imply the (Euclidean) geometric orthogonality of the parametric family of surfaces q_i = constant with the family q_j = constant. [*E.g.*, x' and y' of (3.6) are not (Euclidean) orthogonal coordinates, but their metric does satisfy (2) above.]

Introducing the transformation defined by (1) and (2) in (2.3), we have by analogy with Lamé's transformation[7] of Laplace's equation

$$\Box\phi = (h_1h_2h_3)^{-1}\left[\left(\frac{h_2h_3}{h_1}\phi_{q_1}\right)_{q_1} - \left(\frac{h_3h_1}{h_2}\phi_{q_2}\right)_{q_2} - \left(\frac{h_1h_2}{h_3}\phi_{q_3}\right)_{q_3}\right]. \tag{4.3}$$

It would, of course, be possible to include τ in the transformation, writing

$$(ds)^2 = (dx)^2 - (dy)^2 - (dz)^2 - (d\tau)^2 \tag{4.4}$$

with an obvious extension of (3) for $\Box\phi - \phi_{\tau\tau}$. However, aside from the Lorentz transformation (2.2) we shall include τ only in some rather simple homogeneous transformations, where the introduction of metrical coefficients would appear rather ponderous.

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[5]P. A. Lagerstrom, Jet Propulsion Lab. Rep. 4-36; NACA, T.N. 1685 (1948).

[6]W. D. Hayes, Thesis, Calif. Inst. Tech., Pasadena, Calif., 1947.

*The introduction of the hyperbolic distance in the study of the wave equation is of course not new, having been developed in some generality by Hadamard (ref. 10, *infra;* also ref. 16, 430 ff.). However, the particular problem of separation of variables in hyperbolic space seems to have received little previous consideration.

[7]E. T. Whittaker and G. N. Watson, *Modern analysis*, Macmillan Co., New York, 1948, p. 401.

5. Methods of solution. We shall concern ourselves primarily with obtaining solutions to (2.3), (2.4) and (2.7) by separating variables. It might be thought sufficient to seek solutions of (2.1), on which a considerable literature already exists, but this is not generally the case, due principally to the difficulties associated with moving boundaries. Nevertheless, several interesting results may be so obtained[8], and the approach has the advantage of physical clarity, since T is a direct measure of physical time, whereas τ is not.

An alternative attack based on the existing literature for the wave equation would be to rewrite (2.3) in the form

$$\phi_{\tau\tau} + \phi_{yy} + \phi_{zz} = \phi_{xx} . \tag{5.1}$$

[In view of (1), the remark in ref. 8 that there is no transformation that will fix the coordinate system in the wing and still yield the wave equation seems to require some modification.] This approach has proved quite fruitful in the steady flow case ($\phi_{\tau\tau} = 0$), due both to the physical and mathematical analogies afforded (*e.g.*, von Kármán's acoustic analogy[9]) and, more importantly, the applicability of Hadamard's method[10]. (Nevertheless, in the light of subsequent developments, the most successful of the general methods applicable to the steady flow wing problem appear to have been those of Busemann[11,12,13] and Evvard,[14,15] for which no clearly defined antecedents existed in the classical literature.) While Hadamard's method is not directly applicable to the three dimensional wave equation[16], there exists the even more elegant method of Marcel Riesz[17]. We have not investigated the application of Riesz's method to the unsteady flow problem, but some consideration has been given to this matter by P. A. Lagerstrom[18]. It would appear to be of interest primarily in obtaining a solution to the direct wing problem (ϕ_z specified everywhere on $z = 0$) and in formulating the integral equation for the indirect wing problem (different derivatives of ϕ specified over different parts of $z = 0$).

We turn now to the problem of separation of variables. By analogy with the classical wave equation[19,20] there must exist eleven coordinate systems in which the hyperbolic Helmholtz equation of (2.4) is separable. In general, these systems will differ from their counterparts in Euclidean space, but the cylindrical (with respect to the x axis) systems remain unchanged. Since in each (cylindrical) case the separation of x yields an exponential solution, we consider for these systems the solution of the modified equation (2.7).

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[8]H. Lomax, M. A. Heaslet, and F. B. Fuller, NACA, T.N. 2256 (1950).

[9]Th. von Kármán, J. Aero Sci. **14**, 373 (1947).

[10]J. Hadamard, *Lectures on Cauchy's problem*, Yale Univ. Press, 1923.

[11]A. Busemann, Luftfahrtforschung **12**, 210 (1935).

[12]S. Goldstein and G. N. Ward, Aero. Q. **2**, 39 (1940).

[13]Lagerstrom, l. c. ante.

[14]J. C. Evvard, NACA, T.N. 1382 (1947).

[15]G. N. Ward, Q. J. Mech. and Appl. Math. **2**, 136 (1949).

[16]R. Courant & D. Hilbert, *Methoden der mathematische Physik*, J. Springer, Berlin, 1938, vol. 2, p. 443.

[17]M. Riesz, Acta Math. **81**, 1-218 (1949): see also B. Baker and E. T. Copson, *Huygens' principle* Oxford U. Press, 1950, p. 57.

[18]In the unpublished lectures referred to above.

[19]H. P. Robertson, Math. Ann. **98**, 749 (1938).

[20]L. P. Eisenhart, Ph. Rev. **45**, 427 (1934); Ann. Math. **35**, 284 (1934).

6. Cartesian coordinates. Equation (2.7) is separable in the following cylindrical systems: Cartesian, polar, elliptic and parabolic. The separated solutions in Cartesian coordinates are exponential, and the method of generalization is that of Fourier. The great majority of the literature on the supersonic wing problem uses Cartesian coordinates, and no further discussion is warranted here.

7. Cylindrical polars. Let (ρ, φ) be the polar coordinates defined by

$$y + iz = \rho e^{i\varphi}. \tag{7.1}$$

The corresponding transformation of (2.7) yields

$$\rho(\rho\Phi_\rho)_\rho + \Phi_{\varphi\varphi} - (\lambda\rho)^2\Phi = 0. \tag{7.2}$$

The separated solution of (2), which differs from its classical counterpart only in the sign of λ^2, is given by

$$\Phi = K_\mu(\lambda\rho)e^{i\mu\varphi} \tag{7.3}$$

where K_μ is Macdonald's solution to Bessel's equation of order μ and is dictated (in preference to alternative Bessel functions) by the boundary condition at infinity.

In the case of a rectangular wing edge ($\rho = 0$) with upper and lower surfaces $\varphi = 0$ and π, respectively, the solution

$$\left(\frac{\partial}{\partial y}\right)^{m+1}\Phi^{(m)} = K_{m+1/2}(\lambda\rho)\cos\left[\left(m + \frac{1}{2}\right)\varphi\right] \tag{7.4}$$

is appropriate to the boundary condition

$$\Phi_z^{(m)}|_{z=0^*} \sim y^m. \tag{7.5}$$

(More generally, y^m can be replaced by an m-th order polynomial in y.) and may be used to construct a general solution for the oscillating rectangular wing. This approach has been used by Rott[21] for the special case $m = 0$, following Lamb's treatment of the half plane diffraction problem.[22]

8. Elliptic cylinder coordinates. The well known transformation

$$y + iz = b\cosh(\xi + i\eta) \tag{8.1}$$

yields

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} - (\lambda b)^2(\cosh^2\xi - \cos^2\eta)\Phi = 0 \tag{8.2}$$

which separates into Mathieu equations in both ξ and η. A general solution of (2), subject to the appropriate null condition at infinity, is given by

$$\Phi = Gek_n(\xi, -q)[A_n ce_n(\eta, -q) + B_n se_n(\eta, -q)], \tag{8.3}$$

$$q = \tfrac{1}{4}\lambda^2 b^2, \tag{8.4}$$

where the notation is that of McLachlan[23].

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[21] N. Rott, J. Aero. Sci. **18**, 775 (1951).

[22] H. Lamb, Proc. Lon. Math. Soc. (2), **4**, 190 (1906); *ibid* **8**, 422 (1910); *Hydrodynamics*, Dover Publ., New York, 1945, p. 538.

[23] N. W. McLachlan, *Theory and application of Mathieu functions*, Oxford U. Press, 1947.

These coordinates have been used to obtain a general solution for an oscillating rectangular wing of (in principle) arbitrary aspect ratio[24], following Sieger's solution of the analogous diffraction problem[25]. While the Laplace inversion of the resulting solution in terms of tabulated functions does not appear to be possible, an expansion in powers of λb leads to an asymptotic (in x) expansion for the potential and to integrals (*e.g.*, lift and moment) that are useful for values of the "effective" aspect ratio less than unity. [The solution for a single edge (*vide supra*) suffices to handle the greater than unity case.]

9. Parabolic cylinder coordinates. In this case, we have

$$y + iz = \tfrac{1}{2}(\xi + i\eta)^2, \tag{9.1}$$

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} - \lambda^2(\xi^2 + \eta^2)\Phi = 0 \tag{9.2}$$

and the resulting solution appears in the form

$$\Phi = D_\mu[(2\lambda)^{1/2}\xi]D_{-\mu-1}[(2\lambda)^{1/2}\eta] \tag{9.3}$$

where D_μ is Weber's parabolic cylinder function of order μ in the notation of Whittaker[26].

In practice the manipulation of the D_μ is involved, but, having introduced (ξ, η), it may be possible to write down solutions in terms of more elementary functions. Thus, following Lamb (ref. 22), we find the solution

$$\Phi = e^{\lambda\xi\eta}\left[A + B\int_0^{(\lambda/2)^{1/2}(\xi+\eta)} e^{-\zeta^2}\,d\zeta\right] + e^{-\lambda\xi\eta}\left[C + D\int_0^{(\lambda/2)^{1/2}(\xi-\eta)} e^{-\zeta^2}\,d\zeta\right]. \tag{9.4}$$

Evaluating A, B, C, D, by the imposition of the boundary conditions

$$\Phi_z|_{z=0\pm} = -W(s), \qquad y > 0 \tag{9.5a}$$

$$\Phi = 0, \qquad y \leq 0 \tag{9.5b}$$

together with the null requirement at infinity, we find for the Laplace transform of the potential on the upper surface $(\eta = 0)$ of a rectangular wing

$$\Phi|_{z=0+} = \lambda^{-1}W(s)\ \mathrm{erf}\ [(\lambda y)^{1/2}] \tag{9.6}$$

in agreement with the result obtained by the method suggested in section 7 for the case where the boundary condition on the wing is independent of y.

10. Hyperbolic coordinates. Let r denote the hyperbolic radius, *viz.*

$$r = (x^2 - y^2 - z^2)^{1/2}. \tag{10.1}$$

Then the Euclidean transformation to spherical polar coordinates suggests the analogous transformation

$$x = r\cosh\xi, \qquad y = r\sinh\xi\cos\varphi, \qquad z = r\sinh\xi\sin\varphi. \tag{10.2}$$

If we restrict both r and ξ to be positive and real, only points within the downstream Mach cone $[x > (y^2 + z^2)^{1/2}]$ of the origin are included in the transformation, and the surfaces obtained by holding r, ξ or φ constant are circular hyperboloids (of two sheets,

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[24] J. W. Miles, Aero. Q. **4**, 231 (1953).
[25] B. Sieger, Ann. d. Physik **27**, 626 (1908).
[26] ref. 7, sec. §16.5.

although only the sheet directed along $+x$ is included), circular cones and planes, respectively. The Mach cone itself, being a characteristic surface of $\Box\phi$, has the degenerate specification $r = 0$ *and* $\xi = \infty$. We remark that the tangent planes to the surfaces r = constant and ξ = constant have complementary (rather than perpendicular) slopes.

Introducing the transformation (2) in (4.3) we obtain

$$\Box\phi = (r \sinh \xi)^{-2}[\sinh^2 \xi(r^2\phi_r)_r - \sinh \xi(\sinh \xi\phi_\xi)_\xi - \phi_{\varphi\varphi}] \tag{10.3}$$

and, separating variables, a general solution to (2.4) is given by

$$\phi = r^{-1/2}Z_{\nu+1/2}(\pi r)B_\nu^\mu(\cosh \xi)e^{i\mu\varphi} \tag{10.4}$$

where $Z_{\nu+\frac{1}{2}}$ is a solution to Bessel's equation of order $\nu + \frac{1}{2}$, and B_ν^μ is a solution to the generalized Legendre equation

$$[(1 - z^2)B_z]_z + [\nu(\nu + 1) - \mu^2(1 - z^2)^{-1}]B = 0. \tag{10.5}$$

Solutions of type (4) have been studied by Hayes (ref. 6) for the case of steady flow ($\eta = 0$), where the r dependence reduces to r^ν. The resulting solutions are homogeneous of order ν, or, in the terminology of the day, "generalized conical flows".

An alternative attack on (2.3), after posing the dependence on ξ and φ already found, is to assume the solution to be homogeneous in (τ/r). Thus, it is found that a homogeneous solution of order k is given by

$$\phi = r^k \mid 1 - (\tau/r)^2 \mid^{1/2(k+1)} B_\nu^{k+1}(\tau/r)B_\nu^\mu(\cosh \xi)e^{i\mu\varphi} \tag{10.6}$$

a result that is reminiscent of a homogeneous solution to the wave equation proposed by Bateman[27] and has possible application to transient loading problems.

11. Prolate hyperboloidal coordinates. This system is derived by analogy to the conventional prolate spheroidal set. Modifying the transformation to the latter, we arrive at the new transformation

$$\begin{aligned} x &= \xi\eta - 1 \\ y &= (\xi^2 - 1)^{1/2}(\eta^2 - 1)^{1/2} \cos \varphi \\ z &= (\xi^2 - 1)^{1/2}(\eta^2 - 1)^{1/2} \sin \varphi. \end{aligned} \tag{11.1}$$

As in §10, the entire manifold (ξ, η, φ) is mapped in the downstream Mach cone from the (x, y, z) origin, but to avoid ambiguity we impose the restrictions

$$1 \leq \eta < \xi. \tag{11.2}$$

The surface of revolution ξ = constant, as defined by

$$\frac{(x + 1)^2}{\xi^2} - \frac{(y^2 + z^2)}{(\xi^2 - 1)} = 1 \tag{11.3}$$

is evidently a circular hyperboloid of two sheets (*cf.* §10) directed along the x axis. The same result holds for η = constant, it being necessary only to replace ξ by η in (3). Again, the respective families of surfaces are not orthogonal in the Euclidean sense.

Introducing the transformation (1) in (4.3), we obtain

$$\Box\phi = (\xi^2 - \eta^2)^{-1}\{[(\xi^2 - 1)\phi_\xi]_\xi - [(\eta^2 - 1)\phi_\eta]_\eta - (\xi^2 - 1)^{-1}(\eta^2 - 1)^{-1}\phi_{\varphi\varphi}\}. \tag{11.4}$$

[27] ref. 3, p. 384.

Posing the solution

$$\phi = f(\xi)g(\eta)e^{i\mu\varphi} \tag{11.5}$$

in (2.4), separation of variables yields the Lamé equation

$$[(\xi^2 - 1)f_\xi]_\xi + [\kappa^2\xi^2 - \mu^2(\xi^2 - 1)^{-1} + \nu]f = 0 \tag{11.6}$$

and an identical equation for $g(n)$.

We remark that $f(\xi)$ and $g(\eta)$ need not be the same solution to (6); *i.e.*, they may be Lamé functions of the first and second kinds, respectively, or *vice versa*.

12. Oblate hyperboloidal coordinates. Modifying the conventional transformation to oblate spheroidal coordinates, we write

$$\begin{aligned} x &= \xi\eta, \\ y &= (\xi^2 + 1)^{1/2}(\eta^2 - 1)^{1/2}\cos\varphi, \\ \xi &= (\xi^2 + 1)^{1/2}(\eta^2 - 1)^{1/2}\sin\varphi. \end{aligned} \tag{12.1}$$

In this case, symmetry exists between ξ and $i\eta$, and the surfaces ξ = constant and η = constant, *viz.*

$$\frac{x^2}{\xi^2} - \frac{(y^2 + z^2)}{(\xi^2 + 1)} = 1, \tag{12.2}$$

$$\frac{(y^2 + z^2)}{(\eta^2 - 1)} - \frac{x^2}{\eta^2} = 1 \tag{12.3}$$

are circular hyperboloids of two and one sheets, respectively, directed along the x axis. The restricted range

$$1 \leq \eta < +(\xi^2 + 1)^{1/2} \tag{12.4}$$

includes all points inside the downstream Mach cone, while $\eta > (\xi^2 + 1)^{1/2}$ gives points outside of this cone.

The transformation of the hyperbolic Laplacian yields

$$\Box\phi = (\xi^2 + \eta^2)^{-1}\{[(\xi^2 + 1)\phi_\xi]_\xi - [(\eta^2 - 1)\phi_\eta]_\eta - (\xi^2 + 1)^{-1}(\eta^2 - 1)^{-1}\phi_{\varphi\varphi}\} \tag{12.5}$$

and the resulting separation of (2.4) again leads to Lamé functions.

13. Hyperboloidal coordinates. Modifying the conventional transformation to ellipsoidal coordinates (in the notation of Hobson[28]), we write

$$\begin{aligned} x &= h^{-1}k^{-1}\xi\eta\zeta, \\ y &= h^{-1}(k^2 - h^2)^{-1/2}(\xi^2 - h^2)^{1/2}(\eta^2 - h^2)^{1/2}(\zeta^2 - h^2)^{1/2}, \\ z &= k^{-1}(k^2 - h^2)^{-1/2}(\xi^2 - k^2)^{1/2}(k^2 - \eta^2)^{1/2}(\zeta^2 - k^2)^{1/2}, \end{aligned} \tag{13.1}$$

$$0 < h \leq \eta \leq k \leq \xi < \zeta. \tag{13.2}$$

The restricted range of (2) includes all points within the Mach cone.

The coordinate surfaces are defined by

$$\frac{x^2}{\xi^2} - \frac{y^2}{\xi^2 - h^2} - \frac{z^2}{\xi^2 - k^2} = 1, \tag{13.3}$$

[28] E. W. Hobson, *Spherical and ellipsoidal harmonics*, Cambr. U. Press, 1931, Ch. XI.

where ξ may be replaced by either η or ζ. The ξ and ζ surfaces are elliptic hyperboloids of two sheets directed along the x axis (*i.e.*, the section x = constant yields an ellipse, while either y or z = constant yields a hyperbola), while the η surfaces are elliptic hyperboloids of one sheet directed along the y axis.

The hyperbolic Laplacian is given by

$$\Box\phi = (\zeta^2 - \xi^2)^{-1}(\zeta^2 - \eta^2)^{-1}L_\zeta\{\phi\} - (\zeta^2 - \xi^2)^{-1}(\xi^2 - \eta^2)^{-1}L_\xi\{\phi\} - (\zeta^2 - \eta^2)^{-1}(\xi^2 - \eta^2)^{-1}L_\eta\{\phi\}, \tag{13.4}$$

$$L_\xi\{\phi\} = (\xi^2 - h^2)^{1/2}(\xi^2 - k^2)^{1/2}[(\xi^2 - h^2)^{1/2}(\xi^2 - k^2)^{1/2}\phi_\xi]_\xi\,, \tag{13.5}$$

and the resulting separation of (2.4) yields Lamé functions in each of the three variables. L_ζ is identical with L_ξ, but $(\xi^2 - k^2)^{1/2}$ must be replaced by $(k^2 - \eta^2)^{1/2}$ in L_η.

14. Hyperboloido-conal coordinates. If, in (13.1) *et. seq.* we assume ζ to be very large, so that ζ, $(\zeta^2 - h^2)^{1/2}$ and $(\zeta - k^2)^{1/2}$ each may be replaced by r, we have, in analogy to the spheroconal coordinates of classical potential theory (see ref. 26),

$$\begin{aligned} x &= h^{-1}k^{-1}r\xi\eta, \\ y &= h^{-1}(k^2 - h^2)^{-1/2}r(\xi^2 - h^2)^{1/2}(\eta^2 - h^2)^{1/2}, \\ z &= k^{-1}(k^2 - h^2)^{-1/2}r(\xi^2 - k^2)^{1/2}(k^2 - \eta^2)^{1/2}. \end{aligned} \tag{14.1}$$

These are the hyperboloido-conal coordinates first introduced by Robinson[29] and applied by him to delta wings in both steady and unsteady flow. (Robinson uses various notations in the papers cited and also introduces Jacobian elliptic functions). The surfaces obtained by setting r, ξ or η constant are circular hyperboloids of two sheets directed along the x axis, elliptic cones directed along the x axis and elliptic cones directed along the y axis, respectively. In particular, the surface $\xi = k$ degenerates to the triangular lamina bounded by $y = \pm k^{-1}(k^2 - h^2)^{1/2}x$ and $z = 0\pm$ and lying entirely inside the Mach cone, thereby furnishing the desired separation of variables for the delta wing with subsonic leading edges.

The hyperbolic Laplacian in these coordinates is given by

$$\Box\phi = r^{-2}[(r^2\phi_r)_r - (\xi^2 - \eta^2)^{-1}(L_\xi\{\phi\} + L_\eta\{\phi\})], \tag{14.2}$$

where L_ξ and L_η are given by (13.5). A solution to (2.3) that vanishes on the Mach cone and is regular on the x axis is given by

$$\phi = \psi_n(r, \tau)F_n^m(\xi)E_n^m(\eta), \tag{14.3}$$

where E and F denote Lamé functions of the first and second kind in the notation of Hobson[28], and

$$(r^2\psi_r)_r - n(n + 1)\psi = r^2\psi_{\tau\tau}\,. \tag{14.4}$$

Solutions to (4) may be obtained by comparison with the (r, τ) portions of (10.4) and (10.6), *viz.*

$$\psi_n(\tau, r) = r^{-1/2}Z_{n+1/2}(\kappa r)e^{i\kappa\tau}, \tag{14.5}$$

$$\psi_n(\tau, r) = r^k \mid 1 - (\tau/r)^2 \mid^{1/2(k+1)} B_n^{k+1}(\tau/r). \tag{14.6}$$

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[29]A. Robinson, RAE 2151, ARC 10222 (1946); J. Roy. Aero. Soc. **52**, 735 (1948); Rep. 16, Cranfield Coll. Aero. (1948); Proc. 7th. Inter. Congr. Appl. Mech. **2**, 500 (1948); *cf.* also Haskind and Falkovich, Akad. Nauk. SSSR Prikl. Mat. Mech. **11**, 371 (1947) and Germain and Bader, Recherche Aero. **1949**, 3 (1949).

The homogeneous solutions may be applied to the solutions of the gust loading of a delta wing, but the practical difficulties entailed by the introduction of the Lamé functions are considerable.

A more detailed discussion of the properties of these very useful coordinates is given in the papers cited[29].

15. Parabolic coordinates. A set of coordinates closely related to the classical parabolic coordinates is defined by the analogous transformation [*cf.* (9.1)]

$$x = \tfrac{1}{2}(\xi^2 + \eta^2), \qquad y = \xi\eta \cos\varphi, \qquad z = \xi\eta \sin\varphi. \tag{15.1}$$

While the entire (ξ, η, φ) manifold is mapped inside the downstream Mach cone, we avoid ambiguity by imposing the restriction

$$0 \leq \eta < \xi. \tag{15.2}$$

However, the roles of ξ and η could equally well be reversed by virtue of their symmetry in (1). The surfaces ξ = const. and η = const. are non-orthogonal (in the Euclidean sense) families of paraboloids.

The transformation of the hyperbolic Laplacian yields

$$\Box\phi = (\xi^2 - \eta^2)^{-1}[\xi^{-1}(\xi\phi_\xi)_\xi - \eta^{-1}(\eta\phi_\eta)_\eta] - (\xi\eta)^{-2}\phi_{\varphi\varphi} \tag{15.3}$$

and a general solution to (2.4) is given by

$$\phi = \xi^{-1}\eta^{-1}W_{\nu,1/2\mu}(i\kappa\xi^2)W_{\nu,1/2\mu}(i\kappa\eta^2)e^{i\mu\varphi}, \tag{15.4}$$

where W denotes a Whittaker function[30].

16. Paraboloidal coordinates. A set of coordinates that bears the same relation to the coordinates of §13 as the (relatively little used) paraboloidal to the ellipsoidal coordinates of classical potential theory[31] is given by

$$\begin{aligned} x &= 2^{-1/2}(\xi^2 + \eta^2 + \zeta^2 - h^2 - k^2), \\ y &= (k^2 - h^2)^{-1/2}(\xi^2 - h^2)^{1/2}(\eta^2 - h^2)^{1/2}(\zeta^2 - h^2)^{1/2}, \\ z &= (k^2 - h^2)^{-1/2}(\xi^2 - k^2)^{1/2}(k^2 - \eta^2)^{1/2}(\zeta^2 - k^2)^{1/2}. \end{aligned} \tag{16.1}$$

The ranges of (ξ, η, ζ) are specified by (13.2) for points inside the Mach cone. The coordinate surfaces are given by

$$2x - \frac{y^2}{(\xi^2 - h^2)} - \frac{z^2}{(\xi^2 - k^2)} = \xi^2, \tag{16.2}$$

where ξ may be replaced by either η or ζ in (2). The surfaces ξ, η or ζ = constant are respectively elliptic, hyperbolic and elliptic paraboloids directed along the x axis.

17. Other possibilities. There exist further coordinate systems having orthogonal metrics, as assumed in (4.2), but, in view of the classic investigations of Schroedinger's equation (refs. 19, 20), it does not appear that separation of variables could be achieved in other than the eleven systems enumerated in §6-16.

It should perhaps be pointed out that there exist coordinate systems that do not satisfy (4.2) but that nevertheless may be extremely useful in practice. Thus, the rotation

[30]ref. 7, ch. XVI

[31]J. C. Maxwell, *Treatise on electricity and magnetism*, Oxford Univ. Press, 1904, p. 240.

to "Mach coordinates", (which, of course, are also Cartesian and therefore lead to solutions by separation), furnished by

$$x = 2^{-1/2}(\xi + \eta), \qquad y = 2^{-1/2}(-\xi + \eta), \qquad z = z \tag{17.1}$$

yields

$$(ds)^2 = 2d\xi\, d\eta - (dz)^2, \tag{17.2}$$

$$\Box\phi = 2\phi_{\xi\eta} - \phi_{zz} \, . \tag{17.3}$$

These coordinates have been used to advantage by Evvard[32] in attacking the unsteady flow problem, although it should be remarked that his end results are of questionable validity for time dependences other than linear.*

In the case of steady flow ($\kappa = 0$) there are many more coordinate systems in which (4.2) is a valid representation and $\Box\phi = 0$ can be separated. Bipolar coordinates furnish a cylindrical example, while toroidal coordinates (in which Laplace's equation is separable) can be appropriately modified.

[32] J. C. Evvard, NACA, T.N. 1699 (1948).

*In NACA, T.N. 951 (1950) it is stated that the results of T.N. 1699 are only "approximate."

ADDITION THEOREMS FOR SPHERICAL WAVES*

By

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Abstract. Expansions or "addition theorems" for the spherical wave functions $j_n(kR)P_n^m(\cos\theta)\exp(im\phi)$, $h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi)$, and $h_n^{(2)}(kR)P_n^m(\cos\theta)\exp(im\phi)$, with reference to the origin O, have been obtained in terms of spherical wave functions with reference to the origin O', where O' has the coordinates (r_0, θ_0, ϕ_0) with respect to O.

1. Introduction. When a plane wave is incident on a configuration of spheres, the scattered field may be obtained by a series of successive approximations. The zero-th order approximation is just the sum of the fields scattered by each individual sphere when the excitation of each sphere is taken to be the original plane wave. Higher order approximations take into account contributions to the excitation of a particular sphere from the waves which the remaining spheres scatter. In particular, the first order approximation is the field scattered by the configuration when the excitation field is taken to be the initial plane wave plus the zero-th order approximation to the scattered wave.

In order to carry out this approximation it is necessary to decompose the scattered field of a sphere with center at O into incoming spherical waves for a second sphere with center at O'. Such a decomposition requires an "addition theorem" which expresses a spherical wave with center at O, in terms of spherical waves with center at O'.

Such an expansion or "addition theorem" for cylindrical waves is well known [1]. Using this expansion, Twersky [2] was able to calculate the scattered field obtained by a plane wave striking a configuration of cylinders, taking into account the contributions to the excitations of a particular element by the radiation scattered by the remaining elements. The analogous case of a plane wave striking a configuration of spheres may be treated with the help of the addition theorems we have obtained in this paper.

The first expansion treated in this paper is that for the standing spherical wave $j_n(kR)P_n^m(\cos\theta)e^{im\phi}$ with reference to origin O. This spherical wave is represented as an integral of plane waves over all possible directions. These plane waves are also referred to the origin O. A transformation is then made to obtain the plane waves in terms of the origin O'. In order to evaluate the resulting integral it is necessary to have a formula which expresses the product of two associated Legendre functions in terms of a sum of associated Legendre functions. Once this is obtained by using a formula due to Infeld and Hull, the evaluation of the integral is possible and the final form of the expansion is a sum of spherical waves with reference to the origin O'.

The second expansion treated is that for the outgoing spherical wave $h_n^{(1)}(kR)P_n^m(\cos\theta)e^{im\phi}$ with reference to the origin O. The same procedure is used as for the expansion of $j_n(kR)P_n^m(\cos\theta)e^{im\phi}$. However, in this case, the integral is over plane waves with real and complex directions. This leads to convergence difficulties when the transformation from the origin O to the origin O' is effected. These are discussed in

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Appendix III. The final form of the expansion as a sum of spherical waves with reference to the origin O' is then obtained.

From the first two expansions we then obtain an expansion for $h_n^{(2)}(kR)P_n^m(\cos\theta)e^{im\phi}$ with reference to origin O, in terms of spherical waves referred to origin O' by using the relationship $h_n^{(1)}(kR) + h_n^{(2)}(kR) = 2j_n(kR)$.

We wish to express our thanks to Dr. Victor Twersky who suggested this problem.

2. The expansion for $j_n(kR)P_n^m(\cos\theta)e^{im\phi}$. Consider a point P, which has the spherical coordinates (R, θ, ϕ) with respect to the origin O. A plane wave coming in along the direction $\theta = \alpha$, $\phi = \beta$ can be expressed as follows, in terms of spherical waves with center at O (see [3]):

$$\exp(ikR\cos\gamma) = \sum_{n=0}^{\infty} i^n(2n+1)j_n(kR)P_n(\cos\gamma),$$

or using the addition theorem for Legendre polynomials,

$$\exp(ikR\cos\gamma)$$
$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n(2n+1)j_n(kR)\{n, |m|\}P_n^m(\cos\alpha)P_n^m(\cos\theta)\exp[-im(\beta-\phi)], \tag{1}$$

where we have put

$$\{n, m\} = \frac{(n-m)!}{(n+m)!} \qquad \text{and} \qquad P_n^{-m}(\cos\alpha) = P_n^m(\cos\alpha).$$

Here γ is the angle between the directions (α, β) and (θ, ϕ) so that $\cos\gamma = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\phi-\beta)$. From the expansion (1), an integral representation of elementary spherical wave functions can be obtained. Multiplying both sides of the equation by $P_n^m(\cos\alpha)\exp(im\beta)\sin\alpha$, integrating over α and β, and using the orthogonality properties of the Legendre functions, we obtain the well-known formula (see [3]):

$$j_n(kR)P_n^m(\cos\theta)e^{im\phi} = (i^{-n}/4\pi)\int_0^{2\pi}\int_0^{\pi}\exp(ikR\cos\gamma)P_n^m(\cos\alpha)\exp(im\beta)\sin\alpha\,d\alpha\,d\beta. \tag{2}$$

Introduce a new origin O', where O' has coordinates (r_0, θ_0, ϕ_0) with respect to O (Fig. 1), and let (r, θ', ϕ') be the spherical coordinates of the point P with respect to O'.

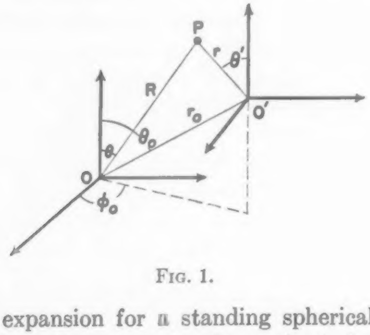

FIG. 1.

We shall now obtain an expansion for a standing spherical wave around O such as $j_n(kR)P_n^m(\cos\theta)\exp(im\phi)$ in terms of standing spherical waves around O' such as $j_\nu(kr)P_\nu^\mu(\cos\theta')\exp(i\mu\phi')$.

First we introduce a set of rectangular coordinates around O and a parallel set around O'. Let (x, y, z) be the coordinates of P relative to O, and let (x', y', z') be the coordinates of P relative to O'. We have

$$\begin{aligned} z &= R\cos\theta, & x &= R\sin\theta\cos\phi, & y &= R\sin\theta\sin\phi, \\ z' &= r\cos\theta', & x' &= r\sin\theta'\cos\phi, & y' &= r\sin\theta'\sin\phi, \\ z &= z' + z_0, & x &= x' + x_0, & y &= y' + y_0. \end{aligned}$$

Now since $\cos\gamma = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\beta - \phi)$, it follows that

$$R\cos\gamma = r[\cos\theta'\cos\alpha + \sin\theta'\cos\phi'\sin\alpha\cos\beta + \sin\theta'\sin\phi'\sin\alpha\sin\beta]$$
$$+ r_0[\cos\theta_0\cos\alpha + \sin\theta_0\cos\phi_0\sin\alpha\cos\beta + \sin\theta_0\sin\phi_0\sin\alpha\sin\beta],$$

which can be written as

$$R\cos\gamma = r\cos\gamma' + r_0\cos\gamma_0. \tag{3}$$

Here γ' is the angle between the direction (α, β) and (θ', ϕ'), while γ_0 is the angle between the direction (α, β) and (θ_0, ϕ_0).

Using (3) in (2) we have

$$_n(kR)P_n^m(\cos\theta)e^{im\phi}$$
$$= \frac{i^{-n}}{4\pi}\int_0^{2\pi}\int_0^{\pi} \exp(ikr\cos\gamma')\exp(ikr_0\cos\gamma_0)P_n^m(\cos\alpha)\exp(im\beta)\sin\alpha\, d\alpha\, d\beta. \tag{4}$$

For the term $\exp(ikr\cos\gamma')$ we use the expansion corresponding to equation (1):

$$\exp(ikr\cos\gamma') = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} i^\nu(2\nu+1)\{\nu, |\mu|\}j_\nu(kr)P_\nu^\mu(\cos\theta')P_\nu^\mu(\cos\alpha)\exp[-i\mu(\phi' - \beta)].$$

It can be shown (see Appendix I) that the above expansion is a uniformly convergent series and hence when it is substituted into equation (4) the order of summation and integration may be interchanged. Therefore we have

$$j_n(kR)P_n^m(\cos\theta)e^{in\phi} = \frac{i^{-n}}{4\pi}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\Big[i^\nu(2\nu+1)\{\nu, |\mu|\}j_\nu(kr)P_\nu^\mu(\cos\theta')$$
$$\cdot e^{-i\mu\phi'}\int_0^{2\pi}\int_0^{\pi}\{\exp(ikr_0\cos\gamma_0)P_n^m(\cos\alpha)P_\nu^\mu(\cos\alpha)\exp[i(m+\mu)\beta]\sin\alpha\}\, d\alpha\, d\beta\Big]. \tag{5}$$

In order to simplify the above equation, it is useful to have a formula expressing the product of two associated Legendre functions in terms of a sum of associated Legendre functions.

Using a formula given by Infeld and Hull [4] for the integral of the product of three Legendre functions, we have

$$P_n^m(\cos\alpha)P_\nu^\mu(\cos\alpha) = \sum_p a(|m|, |\mu| : p, n, \nu)P_p^{m+\mu}(\cos\alpha), \tag{6}$$

where $p = \nu + n, \nu + n - 2, \nu + n - 4, \cdots, \nu - n$, and where, for any coefficient $a(|m|, |\mu|; q, n, \nu)$ we have the relationship

$$a(|m|, |\mu|; q, n, \nu) = \frac{(n+\nu-q-1)!!(2q+1)}{(n+q-\nu)!!(\nu+q-n)!!(q+\nu+n+1)!!}$$
$$\cdot\sum_{j=0}^{p}\exp[\{(n+q-\nu)/2 + |m| + j\}\pi i]\binom{p}{j}\{n, -j-|m|\}\{\nu, |m|+j-q\}.$$

Here $\rho = q - |m| - |\mu|$ and $(s)!! = s(s-2)(s-4) \cdots 2$ or 1, $(0)!! = (-1)!! = 1$, and $\nu - n \leq q \leq \nu + n$.
(In Appendix II some of the $a(|m|, |\mu|; q, n, \nu)$ have been calculated explicitly for some small values of n and m.).

Using equation (6) in equation (5) we have

$$j_n(kR)P_n^m(\cos\theta)e^{im\phi}$$
$$= \frac{i^{-n}}{4\pi}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}\{i^\nu(2\nu+1)\{\nu, |\mu|\}a(|m|, |\mu|; p, n, \nu)j_\nu(kr)P_\nu^\mu(\cos\theta')e^{-i\mu\phi'}I_p^{m,\mu}\} \quad (8)$$

where $I_p^{m,\mu}$ is an integral of the form

$$I_p^{m,\mu} = \int_0^{2\pi}\int_0^{\pi} \exp(ikr_0\cos\gamma_0)P_p^{\mu+m}(\cos\alpha)\exp[i(\mu+m)\beta]\sin\alpha\, d\alpha\, d\beta.$$

It is possible to evaluate these integrals directly [5] (compare equation 4), and we have

$$I_p^{m,\mu} = 4\pi i^p j_p(kr_0)P_p^{\mu+m}(\cos\theta_0)\exp[i(m+\mu)\phi_0]. \quad (9)$$

Substituting equation (9) into (8) we have for the expansion of a spherical wave with respect to one origin, in terms of the spherical waves with respect to another origin, the formula

$$j_n(kR)P_n^m(\cos\theta)e^{im\phi} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}\{i^{\nu+p-n}(2\nu+1)\{\nu, |\mu|\}a(|m|, |\mu|; p, n, \nu)$$
$$\cdot j_p(kr_0)j_\nu(kr)P_p^{\mu+m}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[i(m+\mu)\phi_0]\exp(-i\mu\phi')\}. \quad (10)$$

3. The expansion of $h_n^{(1)}(kR)P_n^m(\cos\theta)e^{im\phi}$ and $h_n^{(2)}(kR)P_n^m(\cos\theta)e^{im\phi}$. In order to obtain an addition theorem for the outgoing spherical wave $h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi)$ we must first get the integral representation for this function. We have the formula [6]

$$\exp(ikr_1)/ikr_1 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2-i\infty}\exp(ikr_1\cos\gamma_1)\sin\alpha\, d\alpha\, d\beta, \quad (11)$$

where r_1, θ_1, ϕ_1 are the coordinates of a point P_1 with respect to the origin O_1. Now, let the point P_1 have spherical coordinates (R, θ, ϕ) with respect to the origin O, and let O_1 have coordinates r^0, ϕ^0, θ^0 with respect to O. From equation (3) we see that

$$R\cos\gamma = r^0\cos\gamma^0 + r_1\cos\gamma_1$$

or

$$\exp(ikr_1\cos\gamma_1) = \exp(-ikr^0\cos\gamma^0)\exp(ikR\cos\gamma) \quad (12)$$

where as before $\cos\gamma = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\beta - \phi)$.

Now from equation (1) we have

$$\exp(-ikr^0\cos\gamma^0)$$
$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^{-n}(2n+1)\{n, |m|\}j_n(kr^0)P_n^m(\cos\theta^0)P_n^m(\cos\alpha)\exp[im(\beta - \phi^0)]. \quad (13)$$

Therefore, using the formula [7]

$$\exp(ikr_1)/ikr_1$$
$$= \sum_{n=1}^{\infty}\sum_{m=-n}^{n}(2n+1)\{n, |m|\}j_n(kr^0)h_n^{(1)}(kR)P_n^m(\cos\theta^0)P_n^m(\cos\theta)\exp[im(\phi - \phi^0)] \quad (14)$$

and equation (12) and (13) in equation (11) and comparing the coefficients of each $j_n(kr^0)P_n^m(\cos\theta^0)$ we find the integral representation to be [8]

$$h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi)$$
$$= \frac{i^{-n}}{2\pi}\int_0^{2\pi}\int_0^{\pi/2-i\infty} \exp(ikR\cos\gamma)P_n^m(\cos\alpha)\exp(im\beta)\sin\alpha\,d\alpha\,d\beta. \tag{15}$$

Now let the point P have spherical coordinates (r', θ', ϕ') with reference to another origin O', where O' has coordinates (r_0, θ_0, ϕ_0) with respect to O. (See Fig. 1). From equations (1) and (3) we see that

$$R\cos\gamma = r\cos\gamma' + r_0\cos\gamma_0$$

and that

$$\exp(ikr\cos\gamma')$$
$$= \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} i^\nu(2\nu+1)\{\nu, |\mu|\}j_\nu(kr)P_\nu^\mu(\cos\theta')P_\nu^\mu(\cos\alpha)\exp[-i\mu(\phi'-\beta)]. \tag{16}$$

When these two relations are substituted into equation (15), it can be shown (see Appendix III) that it is possible to exchange the order of summation and integration provided that $r < r_0$. Therefore we have

$$h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi) = \frac{i^{-n}}{2\pi}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\Big\{i^\nu(2\nu+1)\{\nu, |\mu|\}j_\nu(kr)P_\nu^\mu(\cos\theta')$$
$$\cdot\exp(-i\mu\phi')\int_0^{2\pi}\int_0^{\pi/2-i\infty}\exp(ikr_0\cos\theta_0)P_\nu^\mu(\cos\alpha)P_n^m(\cos\alpha)$$
$$\cdot\exp[i(\mu+m)\beta]\sin\alpha\,d\alpha\,d\beta\Big\} \quad \text{for} \quad r < r_0. \tag{17}$$

Now we have the formula (see equation (6))

$$P_\nu^\mu(\cos\alpha)P_n^m(\cos\alpha) = \sum_p a(|m|, |\mu|; p, n, \nu)P_p^{\mu+m}(\cos\alpha),$$

where $p = \nu+n, \nu+n-2, \cdots, \nu-n$ and the $a(|m||\mu|; p, n, \nu)$ are given by equation (7), substituting the above expression into equation (17) we have

$$h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi) = \frac{i^{-n}}{2\pi}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p i^\nu(2\nu+1)\{\nu, |\mu|\}a(|m|, |\mu|; p, n, \nu)$$
$$\cdot\, j_\nu(kr)P_\nu^\mu(\cos\theta')\exp(-i\mu\phi')K_p^{\mu,m} \tag{18}$$

where the $K_p^{\mu,m}$ are integrals of the form

$$K_p^{\mu,m} = \int_0^{2\pi}\int_0^{\pi/2-i\infty}\exp(ikr_0\cos\theta_0)P_p^{\mu+m}(\cos\alpha)\exp[i(\mu+m)\beta]\sin\alpha\,d\alpha\,d\beta.$$

These integrals can be evaluated directly. Using equation (15) we find that

$$K_p^{\mu,m} = 2\pi i^p h_p^{(1)}(kr_0)P_p^{\mu+m}\exp[i(m+\mu)\phi_0].$$

Therefore, equation (18) becomes

$$h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p\{i^{\nu+p-n}(2\nu+1)\{\nu, |\mu|\}a(|m|, |\mu|; p, n, \nu)$$
$$\cdot j_\nu(kr)h_p^{(1)}(kr_0)P_p^{\mu+m}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[i(m+\mu)\phi_0 - i\mu\phi']\}, \qquad r < r_0.$$

When $r_0 < r$, the resulting expansion would be the above equation with r and r_0 interchanged. Therefore we have

$$h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}\{i^{\nu+p-n}(2\nu+1)\{\nu,|\mu|\}a(|m|,|\mu|;p,n,\nu)$$

$$\cdot\, j_\nu(kr_<)h_p^{(1)}(kr_>)P_p^{\mu+m}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[i(m+\mu)\phi_0]\exp(-i\mu\phi')\}, \tag{19}$$

where $r_< =$ minimum (r, r_0) $r_> =$ maximum (r, r_0). This is the expansion for a spherical wave $h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi)$ about one origin, in terms of spherical waves around another origin.

To find the expansion for $h_n^{(2)}(kR)P_n^m(\cos\theta)\exp(im\phi)$ we use the relationship

$$h_n^{(1)}(kR) + h_n^{(2)}(kR) = 2j_n(kR) \tag{20}$$

and the expansions for $j_n(kR)P_n^m(\cos\theta)\exp(im\phi)$ and $h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi)$. This gives us finally

$$h_n^{(2)}(kR)P_n^m(\cos\theta)\exp(im\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}\{\nu,|\mu|\}(2\nu+1)a(|m|,|\mu|;p,n,\nu)$$

$$\cdot\, j_\nu(kr_<)h_p^{(2)}(kr_>)P_p^{\mu+m}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[i(m+\mu)\phi_0]\exp(-i\mu\phi'). \tag{21}$$

Appendix I.

To show that the expansion

$$\exp(ikr\cos\gamma') = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} i^\nu(2\nu+1)\{\nu,|\mu|\}j_\nu(kr)P_\nu^\mu(\cos\theta')P_\nu^\mu(\cos\alpha)\exp[-i\mu(\phi'-\beta)] \tag{22}$$

is a uniformly convergent series, we first write it in the form

$$\exp(ikr\cos\gamma') = \sum_{\nu=0}^{\infty} i^\nu(2\nu+1)j_\nu(kr)P_\nu(\cos\gamma'). \tag{23}$$

Now we know that for $0 \le \gamma' \le 2\pi$, $|P_\nu(\cos\gamma')| \le 1$, and also $|\exp[-i\mu(\phi'-\beta)]| = 1$. For large values of ν, $|j_\nu(kr)|$ behaves like $(kr/2)^\nu/\nu!$. Hence we have

$$\sum_{\nu=0}^{\infty}|i^\nu(2\nu+1)j_\nu(kr)P_\nu(\cos\gamma')| \le \sum_{\nu=0}^{\infty}(2\nu+1)(kr/2)^\nu/\nu! \tag{24}$$

However, $\sum_{\nu=0}^{\infty}(2\nu+1)(kr/2)^\nu/\nu!$ is a convergent series, and therefore it follows that

$$\sum_{\nu=0}^{\infty} i^\nu(2\nu+1)j_\nu(kr)P_\nu(\cos\gamma') \tag{25}$$

is a uniformly convergent series.

Appendix II.

For general values of ν, n, $|\mu|$ and $|m|$, a formula has been obtained (see equation 7) for the coefficients $a(|m|,|\mu|;p,n,\nu)$. Using this formula, we have calculated the coefficients for some special values of n and m. It is sufficient to evaluate the $a(|m|,|\mu|;p,n,\nu)$ for positive values of m only, since from these the coefficients for negative values of m are readily obtainable and depend upon the definition of $P_n^{-m}(x)$ in terms of $P_n^m(x)$.

We see that for $a(|m|, |\mu|; p, n, \nu) = 0$ unless $p = \nu + 1$ or $\nu - 1$:

$n = 1$	$a(\lvert m\rvert, \lvert\mu\rvert; \nu + 1, 1, \nu)$	$a(\lvert m\rvert, \lvert\mu\rvert; \nu - 1, 1, \nu)$
$m = 0$	$(\nu - \lvert\mu\rvert + 1)/(2\nu + 1)$	$(\nu + \lvert\mu\rvert)/(2\nu + 1)$
$m = 1$	$1/(2\nu + 1)$	$-1/(2\nu + 1)$.

For $n = 2$, the $a(|m|, |\mu|; p, 2, \nu) = 0$ unless $p = \nu + 2, \nu - 2$:

$n = 2$	$a(\lvert m\rvert, \lvert\mu\rvert; \nu + 2, 2, \nu)$	$a(\lvert m\rvert, \lvert\mu\rvert; \nu, 2, \nu)$	$a(\lvert m\rvert, \lvert\mu\rvert; \nu - 2, 2, \nu)$
$m = 0$	$\frac{3}{2}\frac{(\nu - \lvert\mu\rvert + 2)(\nu - \lvert\mu\rvert + 1)}{(2\nu + 3)(2\nu + 1)}$	$\frac{(\nu^2 + \nu - 3\lvert\mu\rvert^2)}{(2\nu + 3)(2\nu - 1)}$	$\frac{3}{2}\frac{(\nu + \lvert\mu\rvert)(\nu + \lvert\mu\rvert - 1)}{(2\nu + 1)(2\nu - 1)}$
$m = 1$	$3\frac{(\nu - \lvert\mu\rvert + 1)}{(2\nu + 3)(2\nu + 1)}$	$\frac{3}{2}\frac{(\nu + \lvert\mu\rvert)}{(2\nu + 3)(2\nu - 1)}$	$-3\frac{(\nu + \lvert\mu\rvert)}{(2\nu + 1)(2\nu - 1)}$
$m = 2$	$\frac{3}{(2\nu + 3)(2\nu + 1)}$	$\frac{-6}{(2\nu + 3)(2\nu - 1)}$	$\frac{3}{(2\nu + 1)(2\nu - 1)}$.

When $n = 3$, $a(|m|, |\mu|; p, 3, \nu) = 0$ unless $p = \nu + 3, \nu + 1, \nu - 1, \nu - 3$.
For $m = 0$

$$a(|m|, |\mu|; \nu + 3, 3, \nu) = \frac{5}{2}\frac{(\nu - |\mu| + 3)(\nu - |\mu| + 2)(\nu - |\mu| + 1)}{(2\nu + 5)(2\nu + 3)(2\nu + 1)}$$

$$a(|m|, |\mu|; \nu + 1, 3, \nu) = \frac{3}{2}\frac{(\nu - |\mu| + 1)(\nu^2 + 2\nu - 5|\mu|^2)}{(2\nu + 5)(2\nu + 1)(2\nu - 1)}$$

$$a(|m|, |\mu|; \nu - 1, 3, \nu) = \frac{3}{2}\frac{(\nu + |\mu|)(\nu^2 - 5|\mu|^2 - 1)}{(2\nu + 3)(2\nu + 1)(2\nu - 3)}$$

$$a(|m|, |\mu|; \nu - 3, 3, \nu) = \frac{5}{2}\frac{(\nu + |\mu|)(\nu + |\mu| - 1)(\nu + |\mu| - 2)}{(2\nu + 1)(2\nu - 1)(2\nu - 3)}.$$

Appendix III.

In order to prove the validity of exchanging the order of summation and integration in equation (17), we first write it in the form

$$h_n^{(1)}(kR)P_n^m(\cos\theta)\exp(im\phi) = \frac{i^{-n}}{2\pi}\lim_{c\to\infty}\int_0^{2\pi}\int_0^{\pi/2-iC}$$
$$\cdot\left\{\left[\sum_{\nu=0}^{\infty} i^\nu(2\nu + 1)j_\nu(kR)P_\nu(\cos\gamma')\right]\exp(ikr_0\cos\gamma_0)P_n^m(\cos\alpha)\exp(im\beta)\sin\alpha\, d\alpha\, d\beta\right\}. \quad (26)$$

However, $|j_\nu(kr)| \le (kr/2)^\nu/\nu!$ and $|P_\nu(\cos\gamma')| \le |\cos\gamma' + (\cos^2\gamma' - 1)^{1/2}|^\nu$; hence

$$\sum_{\nu=0}^{\infty} |i^\nu(2\nu + 1)j_\nu(kR)P_\nu(\cos\gamma')| \le \sum_{\nu=0}^{\infty}\frac{(2\nu + 1)|kr\cos\gamma' + (\cos^2\gamma' - 1)^{1/2}|^\nu}{\nu!2^\nu}$$

where the right hand side is a convergent series, and therefore

$$\sum_{\nu=0}^{\infty} i^{\nu}(2\nu + 1)j_{\nu}(kr)P_{\nu}(\cos \gamma')$$

is a uniformly convergent series for any finite values of r and γ'.

Now, in the finite region containing the segment of the contour of integration included between 0 and $\pi/2 - iC$, $\exp(ikr_0 \cos \gamma_0)P_n^m(\cos \alpha) \sin \alpha \exp(im\beta)$ is bounded, and hence the series

$$\exp(ikr_0 \cos \gamma_0)P_n^m(\cos \alpha) \sin \alpha \exp(im\beta) \sum i^{\nu}(2\nu + 1)j_{\nu}(kr)P_{\nu}(\cos \gamma')$$

is uniformly convergent in this region. Therefore we can write equation (26) as

$$h_n^{(1)}(kR)P_n^m(\cos \theta) \exp(im\phi)$$
$$= \frac{i^{-n}}{2\pi} \lim_{C\to\infty} \left\{ \sum_{\nu=0}^{\infty} i^{\nu}(2\nu + 1)j_{\nu}(kr) \int_0^{2\pi} \int_0^{\pi/2-iC} P_{\nu}(\cos \gamma')P_n^m(\cos \alpha) \right.$$
$$\left. \cdot \exp(ikr_0 \cos \gamma_0) \exp(im\beta) \sin \alpha \, d\alpha \, d\beta \right\}. \tag{27}$$

Equation (27) can then be put in the form

$$h_n^{(1)}(kR)P_n^m(\cos \theta) \exp(im\phi) = \frac{i^{-n}}{2\pi} \sum_{\nu=0}^{\infty} i^{\nu}(2\nu + 1)j_{\nu}(kr) \int_0^{2\pi} \int_0^{\pi/2-i\infty} P_{\nu}(\cos \gamma')P_n^m(\cos \alpha)$$
$$\cdot \exp(ikr_0 \cos \gamma_0) \exp(im\beta) \sin \alpha \, d\alpha \, d\beta - \frac{i^{-n}}{2\pi} \lim_{C\to\infty} \left\{ \sum_{\nu=0}^{\infty} i^{\nu}(2\nu + 1)j_{\nu}(kr) \right.$$
$$\left. \int_0^{2\pi} \int_{\pi/2-iC}^{\pi/2-i\infty} P_{\nu}(\cos \gamma')P_n^m(\cos \alpha) \exp(ikr_0 \cos \gamma_0) \exp(im\beta) \sin \alpha \, d\alpha \, d\beta \right\}. \tag{28}$$

If we can now show that

$$K = \frac{i^{-n}}{2\pi} \lim_{C\to\infty} \left\{ \sum_{\nu=0}^{\infty} i^{\nu}(2\nu + 1)j_{\nu}(kr) \right.$$
$$\left. \cdot \int_0^{2\pi} \int_{\pi/2-iC}^{\pi/2-i\infty} P_{\nu}(\cos \gamma')P_n^m(\cos \alpha) \exp(ikr_0 \cos \gamma_0) \exp(im\beta) \sin \alpha \, d\alpha \, d\beta \right\} = 0, \tag{29}$$

then the proof will be completed.

To prove that $K \to 0$ as $C \to \infty$, we first substitute into equation (29) the expansion

$$P_{\nu}(\cos \gamma') = \sum_{\mu=-\nu}^{\nu} \{\nu, |\mu|\} \cdot P_{\nu}^{\mu}(\cos \theta')P_{\nu}^{\mu}(\cos \alpha) \exp[-i\mu(\phi' - \beta)].$$

(Since this is a finite sum, we may interchange the order of summation and integration.) This gives us

$$K = \frac{i^{-n}}{2\pi} \lim_{C\to\infty} \left\{ \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} i^{\nu}(2\nu + 1)\{\nu, |\mu|\}j_{\nu}(kr)P_{\nu}^{\mu}(\cos \theta') \exp(-i\mu\phi') \right.$$
$$\left. \cdot \int_0^{2\pi} \int_{\pi/2-iC}^{\pi/2-i\infty} P_{\nu}^{\mu}(\cos \alpha)P_n^m(\cos \alpha) \exp(ikr_0 \cos \gamma_0) \exp[i(m + \mu)\beta] \sin \alpha \, d\alpha \, d\beta \right\}. \tag{30}$$

Now, in the (α, β) plane we perform a rotation such that

$$\cos \alpha' = \cos \gamma_0 = \cos \alpha \cos \theta_0 + \sin \alpha \sin \theta_0 \cos (\beta - \phi_0)$$

$$\beta' = \beta - \phi_0 .$$

Then $\cos \alpha = \cos \alpha' \cos \theta_0 + \sin \alpha' \sin \theta_0 \cos (\beta' + \phi_0)$ and the limits of integration remain the same. Hence K becomes

$$K = \frac{i^{-n}}{2\pi} \lim_{C\to\infty} \left\{ \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} i^{\nu}(2\nu + 1)\{\nu, \mid \mu \mid\} j_{\nu}(kr) \exp(-i\mu\phi') P_{\nu}^{\mu}(\cos \theta') \right.$$

$$\cdot \int_0^{2\pi} \int_{\pi/2-iC}^{\pi/2-i\infty} [\exp(ikr_0 \cos \alpha') P_{\nu}^{\mu}(\cos \alpha' \cos \theta_0 + \sin \alpha' \sin \theta_0 \cos (\beta' + \phi_0))$$

$$\cdot P_n^m(\cos \alpha' \cos \theta_0 + \sin \alpha' \sin \theta_0 \cos (\beta' + \phi_0))$$

$$\left. \cdot \exp [i(m + \mu)(\beta' - \phi_0)] \sin \alpha' \, d\alpha' \, d\beta'] \right\}. \tag{31}$$

Now for large values of $\mid z \mid$, we have the following asymptotic form for $P_p^q(z)$, [10]

$$P_p^q(z) \simeq \frac{2^p \Gamma(p + 1/2)(z)^p}{(p - \mid q \mid)! \Gamma(1/2)}.$$

If we let $\alpha' = \pi/2 - i\Psi$, where $C \leq \Psi \leq \infty$, then for very large Ψ, $\cos \alpha'$ behaves like $ie^{\Psi}/2$ and $\sin \alpha'$ behaves like $e^{\Psi}/2$. Hence, using the above asymptotic form we find that

$$\mid P_{\nu}^{\mu}(\cos \alpha' \cos \theta_0 + \sin \alpha' \sin \theta_0 \cos (\beta' + \phi_0)) \mid \simeq \left| \frac{2^{\nu}\Gamma(\nu + 1/2)(e^{\Psi}/2)^{\nu}}{(\nu - \mid \mu \mid)!\Gamma(1/2)} \right|$$

$$\mid P_n^m(\cos \alpha' \cos \theta_0 + \sin \alpha' \sin \theta_0 \cos (\beta' + \theta_0)) \mid \simeq \left| \frac{2^n \Gamma(n + 1/2)(e^{\Psi}/2)^n}{(n - \mid m \mid)!\Gamma(1/2)} \right|.$$

Therefore, on substituting the above expressions into equation (29), and then taking the absolute value of K and integrating over β', we have

$$\mid K \mid \leq \left| \frac{i^{-n}\Gamma(n + 1/2)}{2\pi\Gamma(1/2)\Gamma(1/2)(n - \mid m \mid)!} \right|$$

$$\cdot \lim_{C\to\infty} \left\{ \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left| \frac{i^{\nu}(2\nu + 1)\Gamma(\nu + 1/2)}{(\nu + \mid \mu \mid)!} j_{\nu}(kr) P_{\nu}^{\mu}(\cos \theta') \exp(-i\mu\phi') \right| \pi \right.$$

$$\left. \cdot \int_C^{\infty} (e^{\Psi})^{\nu+n} \exp(-e^{\Psi}kr_0/2) e^{\Psi} \, d\Psi \right\}. \tag{32}$$

The integral in equation (30) is now a portion of the real axis. Letting $t = e^{\Psi}$ the integral becomes

$$\int_D^{\infty} t^{\nu+n} \exp(-e^{\Psi}kr_0/2) e^{\Psi} \, d\Psi = \int_D^{\infty} t^{\nu+n} \exp(-tkr_0/2) \, dt \tag{33}$$

where $D = e^C$ and therefore as $C \to \infty$, $D \to \infty$. Integrating equation (33) by parts, it becomes

$$\int_D^\infty t^{\nu+n} \exp(-tkr_0/2)\, dt$$
$$= \left\{ -\frac{2t^{\nu+n} \exp(-tkr_0/2)}{kr_0} \Big|_D^\infty + \frac{2(\nu+n)}{kr_0} \int_D^\infty t^{\nu+n-1} \exp(-tkr_0/2)\, dt \right\}. \tag{34}$$

Therefore

$$\int_C^\infty (e^\Psi)^{\nu+n} \exp(-e^\Psi kr_0/2) e^\Psi\, d\Psi \le \frac{2D^{\nu+n} \exp(-Dkr_0/2)}{kr_0}, \tag{35}$$

and using this inequality with equation (32), the result is

$$| K | \le \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(1/2)(n - | m |)!kr_0} \lim_{D\to\infty} D^n \exp(-kr_0 D/2)$$
$$\cdot \sum_{\nu=0}^\infty \sum_{\mu=0}^\infty \frac{2(\nu+1/2)\Gamma(\nu+1/2)}{(\nu + | \mu |)!} | j_\nu(kr) | \, | P_\nu^\mu(\cos\theta') | D^\nu. \tag{36}$$

However,

$$| P_\nu^\mu(\cos\theta') | \le \frac{2}{\pi^{1/2}(2\sin\theta')^{1/2}} \frac{(\nu + | \mu |)!}{\Gamma(\nu+3/2)}$$

and

$$| j_\nu(kr) | \simeq \frac{(kr/2)^\nu}{\nu!}$$

for large values of ν. Putting these values for $| P_\nu^\mu(\cos\theta') |$ and $| j_\nu(kr) |$ into equation (34) and writing $(\nu + 1/2)\Gamma(\nu + 1/2)$ as $\Gamma(\nu + 3/2)$, we have

$$| K | \le \frac{\Gamma(n+1/2)}{\pi^{3/2}kr_0(2\sin\theta')^{1/2}(n - | m |)!} \lim_{D\to\infty} D^n \exp(-kr_0 D/2) \sum_{\nu=0}^\infty \frac{(krD/2)^\nu}{\nu!}. \tag{37}$$

However

$$\sum_{\nu=0}^\infty \frac{(krD/2)^\nu}{\nu!} = \exp(krD/2),$$

and therefore equation (37) becomes

$$| K | \le A \lim_{D\to\infty} D^n \exp | D(r - r_0)k/2 | \tag{38}$$

where A is the constant

$$\frac{\Gamma(n+1/2)}{\pi^{3/2}kr_0(2\sin\theta')^{1/2}(n - | m |)!}.$$

If we now let $D \to \infty$ we see that $| K | \to 0$ provided that $r < r_0$. Therefore equation (28) becomes

$$h_n^{(1)}(kR)P_n^m(\cos\theta) \exp(im\phi) = \frac{i^{-n}}{2\pi} \sum_{\nu=0}^\infty i^\nu(2\nu+1) j_\nu(kr)$$
$$\cdot \int_0^{2\pi} \int_0^{\pi/2 - i\infty} P(\cos\gamma')P_n^m(\cos\alpha) \exp(ikr_0\cos\gamma_0) \exp(im\beta) \sin\alpha\, d\alpha\, d\beta \text{ when } r < r_0 \tag{39}$$

and when $P_\nu(\cos\gamma')$ is expanded in terms of $P_\nu^\mu(\cos\alpha)P_\nu^\mu(\cos\theta')$ we have equation (16).

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TWO DIMENSIONAL SOURCE FLOW OF A VISCOUS FLUID*

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Summary. The steady two-dimensional source-type flow of a viscous heat-conducting perfect gas is investigated. The solutions of physical significance all contain shocks, and bounds are given for the shock-thickness in terms of the shock-strength and the Reynolds number of the flow. It is found that the entropy rises to a maximum within the shock, and this maximum does not disappear even when the viscosity tends to zero.

Introduction. The main part of this investigation is concerned with two-dimensional source flow similar to the flow in a divergent channel with straight walls, for instance, in a nozzle or a diffuser, when boundary layers are neglected. If the fluid is assumed inviscid, no fundamental length exists, but if it is viscous a Reynolds number R characterizing the flow is provided by the ratio of the mass flow (per unit length normal to the plane of the flow) to the viscosity (cf. eq. (1.14)). In ordinary supersonic wind-tunnel nozzles this Reynolds number is of the order of 10^7, but in low-density, hypersonic wind tunnel nozzles (for which indeed the conical shape is being increasingly favoured) it can be of very much smaller order, and deviations of the flow from that obtained in the limit $R \to \infty$ may be of some interest. The problem has also been considered by Sakurai [1], who derived an equation similar to (1.41), and sketched the solution curves for $R \approx 20$ (see **1.4**).

The problem is, moreover, of theoretical interest since the corresponding problem for an inviscid fluid is one of the few for which an exact solution containing a limit line has been found [2]. This limit line is the sonic circle, and its exterior is doubly covered by the velocity field (**1.2**). The subsonic branch of the solution has a stagnation point at infinity and we aim to find the corresponding solutions for a viscous fluid.

The energy equation is integrated once to give two first order simultaneous differential equations for V and θ as functions of w, where V is essentially the velocity gradient, and w and θ are respectively the speed and temperature in dimensionless notation (**1.1** and **1.3**). All the solutions which have a stagnation point at infinity are shown to have formal asymptotic expansions for V and θ, in this neighbourhood, which agree to the first order with the inviscid solution when $R \to \infty$ (**1.3**).

The simultaneous differential equations for V and θ are of the singular perturbation type. The highest derivatives are multiplied by a small parameter, namely R^{-1}, and for a first approximation to equations of this type, the small parameter is taken to be zero. It is fairly obvious that as long as the highest derivatives remain 'small', there are solutions of the full equations which differ little from the solutions of the lower-order approximate equations. But it cannot be expected that the full solutions will converge *uniformly* to the approximate solutions over the whole flow field, as the parameter tends to zero. The boundary layer is a case in point, and in general, one may expect regions where the limiting solution is quite different from the inviscid approximation.

For a particular value of the Prandtl number σ, the energy equation can be integrated a second time, and the simultaneous equations are then reduced to one first

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order, non-linear differential equation for V in terms of w, which is still of the perturbed type (1.4 and eq. 6.5). This value of σ is $3/4 + 0(R^{-1})$, and as $\sigma \approx 0.72$ for air, the results obtained below should be significant.

In sec. 2, 3 and 4, the solution curves of this differential equation are discussed in the w-V plane by semi-geometrical methods for fixed, large R. Simpler curves are given which provide bounds enclosing the solution curves. In the limit as $R \to \infty$, the singularity found is not a limit line, but an *ideal shock* joining parts of inviscid solutions. A limit line does not occur even for the exceptional solutions, physically unrealisable, which do not contain a shock.

A full discussion is given for the case of constant viscosity. For the case of a viscosity proportional to the absolute temperature, the method of solution is outlined, and the results are summarised in sec. 6. In practice the variation of viscosity with temperature lies between these two extremes. There is no qualitative difference between the two cases as regards the shock formation. However, in the case of variable viscosity, the absolute temperature is automatically positive throughout the flow if it is positive at any one point of it, whereas in the case of constant viscosity there are solutions violating this requirement which have to be discarded on physical grounds.

1. The fundamental equations.

1.1. In this section, the equations for the steady flow of a perfect gas are first considered in general, in order to derive a first integral of the energy equation. The equations are then specialised to the case of purely radial, two-dimensional flow.

Let x_1, x_2, x_3 be a right-handed system of Cartesian coordinates, let v_i be the component of velocity in the direction of x_i increasing, and let p, ρ, T, μ, λ, R, C_v, C_p and a, denote respectively the pressure, density, absolute temperature, viscosity, heat-conductivity, gas constant, specific heat at constant volume, specific heat at constant pressure, and local speed of sound $((\partial p/\partial \rho)_S)$. The equations for the conservation of mass, momentum and energy are, with the usual summation convention,

$$\frac{\partial}{\partial x_i}(\rho v_i) = 0, \tag{1.1}$$

$$\rho v_i \frac{\partial v_i}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(p + \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\right)\delta_{ij} + 2\frac{\partial}{\partial x_j}(\mu e_{ij}), \tag{1.2}$$

and

$$\rho v_i \frac{\partial}{\partial x_i}(C_v T) + p\frac{\partial v_k}{\partial x_k} = \mu\left\{2e_{ij}e_{ij} - \frac{2}{3}\frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right\} + \frac{\partial}{\partial x_i}\left(\lambda\frac{\partial T}{\partial x_i}\right), \tag{1.3}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{1.4}$$

and δ_{ij} is the Kronecker delta. Multiplication of (1.2) by v_i and addition to (1.3) gives

$$\frac{\partial}{\partial x_i}\left\{\rho v_i C_v T + \frac{1}{2}\rho v_i v_j^2 + p v_i + \frac{2}{3}\mu v_i\frac{\partial v_j}{\partial x_j} - 2\mu v_j e_{ij} - \lambda\frac{\partial T}{\partial x_i}\right\} = 0,$$

and so

$$\rho v_i C_v T + \frac{1}{2}\rho v_i v_j^2 + p v_i + \frac{2}{3}\mu v_i\frac{\partial v_j}{\partial x_j} - 2\mu v_j e_{ij} - \lambda\frac{\partial T}{\partial x_i} = \epsilon_{ijk}\frac{\partial A_k}{\partial x_j}, \tag{1.5}$$

where the A_i are unknown functions of the x_i.

The flows investigated in the following are those which are independent of x_3, and in the (x_1, x_2)-plane, depend only on the distance, r, from the origin. The only non-vanishing velocity component is that in the direction of r increasing, denoted by u. Referred to polar coordinates, (r, φ) in the (x_1, x_2) plane, (1.5) then yields

$$\rho u\left(C_v T + \frac{p}{\rho} + \frac{1}{2}u^2\right) + \frac{2}{3}\mu u\left(\frac{du}{dr} + \frac{u}{r}\right) - 2\mu u\frac{du}{dr} - \lambda\frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial A_3}{\partial \varphi}, \tag{1.6}$$

and

$$0 = \frac{\partial A_3}{\partial r}. \tag{1.7}$$

Hence A_3 is a function of φ only, and $\partial A_3/\partial\varphi$ must be a constant.

From equation (1.1)

$$\rho u r = \kappa \text{ (a constant)}, \tag{1.8}$$

and (1.2) gives

$$\rho u\frac{du}{dr} + \frac{dp}{dr} = \frac{4}{3}\frac{d}{dr}\left(\mu\frac{du}{dr}\right) + \frac{2\mu}{r}\frac{du}{dr} - \frac{2\mu u}{r^2} - \frac{2}{3}\frac{d}{dr}\left(\frac{\mu u}{r}\right). \tag{1.9}$$

With the help of the equation of state of the gas,

$$p = \mathrm{R}\rho T, \tag{1.10}$$

(1.6) may be written

$$\rho u r\left(C_p T + \frac{1}{2}u^2\right) + \frac{2}{3}\mu u\left(r\frac{du}{dr} + u\right) - 2\mu u r\frac{du}{dr} - \lambda r\frac{dT}{dr} = C'. \tag{1.11}$$

Equations (1.8) to (1.11) govern the steady, two-dimensional, purely radial flow of a perfect gas.

These equations are put into non-dimensional form by the substitutions

$$w = \left(\frac{\gamma+1}{2}\right)^{1/2}\frac{u}{a_0} = \left(\frac{\gamma+1}{2}\right)^{1/2}\mathrm{R}^{-1/2}T_0^{-1/2}u, \qquad \theta = \frac{T}{T_0}, \qquad \xi = \log r. \tag{1.12}$$

The point at infinity will be taken to be a stagnation point, and T_0 and a_0 denote the corresponding temperature and speed of sound, respectively; $\gamma = C_p/C_v$. The Prandtl number σ is defined by

$$\sigma = \frac{C_p\mu}{\lambda}, \tag{1.13}$$

and the only remaining dimensionless parameter is

$$R = \frac{3\kappa}{\mu}\frac{\gamma+1}{2\gamma}, \tag{1.14}$$

which may be regarded as the Reynolds number of the flow.

Elimination of p and ρ from equations (1.8) to (1.11), and use of (1.12) to (1.14) leads to the two equations

$$(\beta+1)w^2w' + w\theta' - w\theta - \theta w' = \frac{4}{R}w^2(w'' - w) + 2w^2(2w' - w)\frac{d}{d\xi}\left(\frac{1}{R}\right), \tag{1.15}$$

$$\theta + \beta\left\{1 + \frac{4}{R(1+\beta)}\right\}w^2 - \frac{3}{R(1+\beta)\sigma}\theta' - \frac{8\beta}{R(1+\beta)}ww' = C \text{ (a constant)}, \tag{1.16}$$

where

$$\beta = (\gamma - 1)/(\gamma + 1) \tag{1.17}$$

and a dash denotes differentiation with respect to ξ.

1.2. The "inviscid solution." The solution for source flow in an inviscid perfect gas is obtained by putting $R = \infty$ in the above equations, without any enquiry as to the validity of such a step. It will be seen later that some of the terms which have a factor of order R^{-1}, are themselves large of order R, so that even when R becomes infinite a finite contribution remains. If this latter possibility is ignored, the equations reduce to

$$w\{(1 + \beta)ww' + \theta' - \theta\} - \theta w' = 0 \tag{1.18}$$

and

$$\theta + \beta w^2 = 1, \tag{1.19}$$

since $\theta = 1$ at the stagnation point at infinity by definition.

When θ is eliminated from (1.18) and (1.19) the equation

$$w' = -w \frac{(1 - \beta w^2)}{(1 - w^2)} \tag{1.20}$$

is obtained. Its solution is

$$r = e^{\xi} = \frac{\kappa(1 - \beta)^{-1/2}}{a_0 \rho_0 w} \cdot (1 - \beta w^2)^{(\beta - 1)/(2\beta)}, \tag{1.21}$$

where ρ_0 is the stagnation density at $r = \infty$.

The graph of $w(r)$ is given in Fig. 1. The 'sonic speed' (which is the fluid speed at

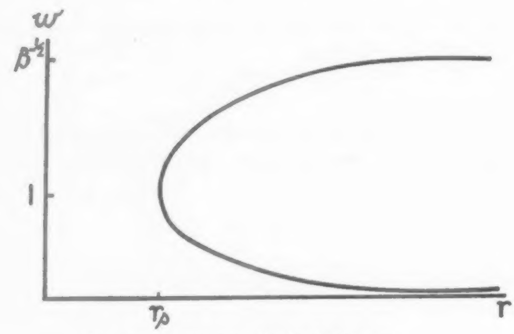

Fig. 1. w versus r for inviscid gas.

which the fluid speed equals the local speed of sound) corresponds to $w = 1$, and the maximum speed attainable, at which the temperature falls to absolute zero, corresponds to $w = \beta^{-1/2}$. If r_s is the value of r for which $w = 1$, (1.20) gives no solution for $r < r_s$, but for $r > r_s$ there are two branches of the $w - r$ curve, one tending to zero, and the other tending to $w = \beta^{-1/2}$ as r tends to infinity. Thus for any r greater than r_s there are two possible values of w, one representing a supersonic speed, and the other a subsonic speed. The streamlines are radial and, in fact, are cusped at the sonic circle, which is a limit line (of a rather special type).

It is intended now to use this solution as a guide to a study of the solutions of the equations (1.15) and (1.16), inasmuch as it is expected that source flow, even in a viscous, heat-conducting fluid, should lead to a stagnation point at $r = \infty$, and that viscous effects should become comparatively unimportant there. But a limit line cannot exist in the real fluid, and so, if solutions of (1.15) and (1.16) are sought which behave like the solution of (1.20) for large r and small w, their continuation backward in r should show what phenomena are to be expected in a real fluid when inviscid theory predicts a limit line. To preserve the correspondence between the solutions for large r, the constant C in (1.16) is taken to be unity.

1.3. The solution at large distances from the source, for large, but finite, values of R. In this section, μ is taken to be a constant, which leads to a considerable simplification at this stage. When the complete solution is discussed later, however, the case of μ varying with temperature will also be considered. It should be noticed here that the ratio of the specific heats, γ, is assumed constant throughout, and the Prandtl number, σ, is assumed constant when the viscosity is constant.

It is possible to eliminate any explicit dependence on ξ from the equations (1.15) and (1.16) by the substitution

$$V = -2\frac{dw}{d\xi} = -2r\frac{dw}{dr}, \tag{1.22}$$

which permits us to reduce these equations to two simultaneous first order equations in V, θ and w, namely

$$\frac{1}{R}w^2V\frac{dV}{dw} = -\frac{1}{2}(1+\beta)w^2V - \frac{1}{2}Vw\frac{d\theta}{dw} - w\theta + \frac{1}{2}\theta V + \frac{4w^3}{R}, \tag{1.23}$$

and

$$\theta - 1 + \beta\left\{1 + \frac{4}{R(1+\beta)}\right\}w^2 + \frac{3}{2R\sigma(1+\beta)}wV = 0. \tag{1.24}$$

The variable V is closely related to the fluid acceleration, and satisfies a certain non-linear differential equation which is obtained by eliminating θ from (1.23) and (1.24).

The equations (1.23) and (1.24) respectively may be put into the forms

$$\begin{aligned}\frac{d}{dw}(V-2w) - \frac{R}{2w^2V}(V-2w) &= -\frac{R}{2}\left\{1+\beta-\frac{8\beta}{(1+\beta)R}+\frac{4}{R}\right\} + \frac{R}{2w^2}(\theta-1)\\ &\quad + R\left\{\beta + \frac{1+2\beta}{1+\beta}\frac{4}{R}\right\}\frac{w}{V} - \frac{R}{2w}\left\{1-\frac{3}{(1+\beta)R\sigma}\right\}\frac{d\theta}{dw}\\ &= f\left(V, \theta, \frac{d\theta}{dw}, w\right)\end{aligned} \tag{1.25}$$

and

$$\begin{aligned}\frac{d}{dw}(\theta-1) + \frac{2R\sigma}{3V}(1+\beta)(\theta-1) &= -\frac{2\beta R\sigma}{3V}(1+\beta)\left\{1+\frac{4}{R(1+\beta)}\right\}w^2 - \frac{8\beta\sigma}{3}w\\ &= g(V, w).\end{aligned} \tag{1.26}$$

If $z(w)$, $\zeta(w)$ are defined by

$$z(w) = -\int^{w} \frac{R}{2w^2 V}\, dw, \tag{1.27}$$

$$\zeta(w) = \int^{w} \frac{2R\sigma}{3V}(1 + \beta)\, dw, \tag{1.28}$$

then equations (1.25) and (1.26) have the formal solutions

$$V - 2w = e^{-z} \int e^{z} f \cdot dw + C_1 e^{-z} \tag{1.29}$$

and

$$\theta - 1 = e^{-\zeta} \int e^{\zeta} g\, dw + C_2 e^{-\zeta}. \tag{1.30}$$

These integral equations may be solved iteratively in the neighbourhood of $w = V = 0$, starting with the first approximation

$$V = 2w \tag{1.31}$$

suggested by the inviscid solution. If $(\theta - 1)$ is to be finite near the stagnation point, C_2 must be zero, and the first approximation to (1.30) is

$$\theta - 1 = -\beta\sigma \frac{12 + R(1 + \beta)}{6 + R\sigma(1 + \beta)} w^2; \tag{1.32}$$

after substitution of this expression in $f(v, \theta, w)$, (1.29) gives the second approximation

$$V - 2w = 2\left\{\frac{(1 - \beta^2)R\sigma + 6(1 + \beta) - 24\beta\sigma}{(1 + \beta)R\sigma + 6}\right\} w^3 + C_1 e^{-z}. \tag{1.33}$$

This process can be repeated as many times as desired to obtain higher approximations, and gives the asymptotic solution for V and θ near $w = V = 0$. The coefficients of the powers of w in the series so obtained agree with those found for the 'inviscid' solution, except for terms of order R^{-1}. However, the former series are divergent, as the term of order R^{-1} in the coefficient of w^n is unbounded as $n \to \infty$. Also there is an infinite number of solutions having the same sort of behaviour near $w = V = 0$, due to the presence of the term $C_1 e^{-z}$ in (1.33), where C_1 is arbitrary. This is of course due to the fact that the differential equations leading to these expansions are of higher order than the 'inviscid' differential equations. The term $C_1 e^{-z}$ is asymptotically of the form

$$C_1' \exp\left\{-\frac{1}{8} R w^{-2} - \frac{1}{4}(1 + \beta) R \log w\right\},$$

and so is very small if w is small enough, whatever the value of C_1.

Thus there is an infinite number of solutions for the flow of the viscous heat-conducting gas, which, in the neighbourhood of the stagnation point at infinity, are asymptotically equal to the solution for the inviscid gas when the viscosity and the heat-conductivity tend to zero. It is very difficult to investigate the solutions of (1.23) and (1.24) in other parts of the flow field, but it is found that there is a particular value of σ which enables the problem to be reduced to that of the solution of a first order non-

linear differential equation. This equation is still of the singular perturbation type, but its solutions can be discussed, after some trouble, and it is with this problem that we shall be concerned hereafter.

1.4. The equation (1.24) may be written

$$\theta - 1 + \beta\left\{1 + \frac{4}{R(1+\beta)}\right\}w^2 - \frac{3}{(1+\beta)R\sigma}\frac{d}{d\xi}(\theta - 1) - \frac{4\beta}{R(1+\beta)}\frac{d}{d\xi}w^2 = 0, \tag{1.34}$$

and this equation, as it stands, can be integrated when

$$\sigma = \sigma_0 = \frac{3}{4} + \frac{3}{R(1+\beta)}\,. \tag{1.35}$$

For, when E is defined by

$$E = \theta - 1 + \beta\left\{1 + \frac{4}{R(1+\beta)}\right\}w^2, \tag{1.36}$$

the equation takes the form

$$E - \frac{4}{R(1+\beta) + 4}\frac{dE}{d\xi} = 0, \tag{1.37}$$

which has the general solution

$$E = A\ \exp\left\{\left[\frac{1}{4}(1+\beta)R + 1\right]\xi\right\} \tag{1.38}$$

where A is an arbitrary constant. Now, E must tend to zero at the stagnation point at infinity, and so $A = 0$ and $E \equiv 0$. In fact, E is effectively the difference between the total energy, per unit mass, of the fluid and the value of this total energy at the stagnation point, and (1.38) shows that $E \to \pm\infty$ at this stagnation point unless $E \equiv 0$. Therefore, when σ has the value given by (1.35),

$$\theta = 1 - \beta\left\{1 + \frac{4}{R(1+\beta)}\right\}w^2. \tag{1.39}$$

This permits us to reduce the problem to a first order differential equation for V as a function of w.

The value chosen for σ is actually not far from the truth, as experimental values for air are in the neighbourhood of 0.72, and when R is large, the value chosen will be close to 0.75. From the form of (1.15) and (1.16), (with constant R) it follows from [5] that the solutions are continuous in σ, and in fact have continuous derivatives of any order with respect to σ in $w > 0$, $-\infty < \sigma < \infty$. In the neighbourhood of $w = 0$ the variation with σ can be seen from (1.33). Thus the only effect of choosing the above particular value of σ is to simplify the equations rather than materially affect their solutions, and we expect that the solutions investigated hereafter will be representative of—and, in fact, very close to—the actual flows which would occur in air under the same boundary conditions. In [1], Sakurai fits values of σ_0 to experimental values for various gases by choosing appropriate values of R, (negative for air), and sketches the solution curves for $\sigma_0 = 0.88$ ($R \approx 20$), Meyer's theoretical value.

With the help of equation (1.39), θ and its derivatives are eliminated from equation (1.23) to give

$$\frac{w^2V}{R}\frac{dV}{dw} = \frac{1}{2}V\left\{1 - \left(1 - \frac{4\beta}{R(1+\beta)}\right)w^2\right\} - w\left\{1 - \left(\beta + \frac{4}{R}\frac{1+2\beta}{1+\beta}\right)w^2\right\}. \tag{1.40}$$

For convenience in the subsequent algebra, terms which are genuinely of order R^{-1} compared with terms of order 1, are now neglected on the right-hand side of (1.40). That is, in each of the curly brackets, the coefficients of w^2 are replaced by 1 and β respectively. Thus, the equation to be investigated is

$$\frac{w^2 V}{R}\frac{dV}{dw} = \frac{1}{2} V(1 - w^2) - w(1 - \beta w^2). \tag{1.41}$$

The methods hereafter applied to (1.41) are quite applicable to (1.40), but the algebra becomes much more involved, merely because of the unwieldy forms of the two coefficients of w^2. The difference between the two equations is very small for large R, and physically the difference is that in equation (1.40) the sonic speed and maximum speed vary slightly with R, while in equation (1.41) they are fixed at their inviscid values. The variation in w between the two is only of order R^{-1}, while in the subsequent discussion the most significant variations in w are of order $R^{-1/2}$.

2. Properties of the solution curves.

Although equation (1.41) is of a simple appearance, it cannot be integrated to find closed solutions, so that the discussion requires indirect methods. In the first place, in order to investigate the form of the solutions we use the standard techniques of curve sketching to discuss the behaviour of the solution curves in the w-V plane, and to this end the behaviour of the curves of zero slope and zero curvature is established in **2.1**, while **2.2** gives some of the more elementary properties of the solution curves.

2.1. Let C_1 be the curve on which dV/dw is zero in equation (1.41), that is, the curve given by

$$V = f(w) = 2w\,\frac{1 - \beta w^2}{1 - w^2}, \tag{2.1}$$

which is also the inviscid solution curve (1.20).

Let C_2 be the curve on which d^2V/dw^2 is zero in equation (1.41), that is, the curve given by

$$V^3 - w(1 + \beta w^2)V^2 - \tfrac{1}{2}R(1 - w^2)(1 - \beta w^2)V + Rw(1 - \beta w^2)^2 = 0. \tag{2.2}$$

It has the following properties.

(i) When $w = 0$ (stagnation point), $V = 0$ or $\pm(R/2)^{1/2}$.

(ii) When $w = 1$ (sonic speed),

$$V = -(1 - \beta)^{2/3}R^{1/3}\{1 + O(R^{-1/3})\}. \tag{2.3}$$

(iii) When $w = \beta^{-1/2}$ (maximum speed), $V^2 = 0$ and $V = 2\beta^{-1/2}$. At $V^2 = 0$, there is a double point, the curve having there the slopes $-\frac{1}{2}(1 - \beta)R\{1 + O(R^{-1})\}$ and $4\beta/(1 - \beta)\{1 + O(R^{-1})\}$.

(iv) For $w \gg R^{1/2}$ there are three branches of the curve, given asymptotically by

$$V \sim 2\beta w,$$

$$V \sim -\tfrac{1}{2}Rw,$$

$$V \sim \beta w^3.$$

(v) The curve C_2 has infinite slope at (w_m, V_m) where

$$w_m \sim 1 - \frac{3}{2^{2/3}}(1 - \beta)^{1/3}R^{-1/3},$$

$$V_m \sim 2^{-1/6}(1 - \beta)^{2/3}R^{1/3}. \tag{2.4}$$

(vi) Provided that

$$| 1 - w^2 |^{-1}R^{-1/3} = o(1) \tag{2.5}$$

there is a branch of C_2 given by

$$V = g_1(w) = f(w)\{1 + O(R^{-1}(1 - w^2)^{-3})\} \tag{2.6}$$

which lies close to C_1 in the range $0 \leq w < \infty$. The condition (2.5) means that the difference between w and 1 is of greater order than $R^{-1/3}$ as $R \to \infty$; that is, it excludes a small range of speeds near the sonic speed. In the range $0 \leq w < w_m$, with the above small neighbourhood of $w = 1$ again excluded, there are two other branches given by

$$V = g_2(w) = \left\{\frac{R}{2}(1 - w^2)(1 - \beta w^2)\right\}^{1/2}\{1 + o(1)\}, \tag{2.7}$$

and

$$V = g_3(w) = -\left\{\frac{R}{2}(1 - w^2)(1 - \beta w^2)\right\}^{1/2}\{1 + o(1)\}.$$

In the range $1 < w < \beta^{-1/2}$, there is only one branch $V = g_1(w)$ which is actually the continuation of $V = g_3(w)$.

(vii) The curves C_1 and C_2 intersect only at $w = 0$ and $w = \beta^{-1/2}$. For $w \ll 1$,

$$f(w) \sim 2w + 2(1 - \beta)w^3 + 2(1 - \beta)w^5 + 2(1 - \beta)w^7 + \cdots \tag{2.8}$$

and

$$\begin{aligned} g_1(w) \sim 2w &+ \{2(1 - \beta) + 8R^{-1}\}w^3 \\ &+ \{2(1 - \beta) + 8(5 - 4\beta)R^{-1} + 128R^{-2}\}w^5 + \{2(1 - \beta) \\ &+ 8(14 - 16\beta + 3\beta^2)R^{-1} + 64(17 - 12\beta)R^{-2} + 2688R^{-3}\}w^7 + \cdots. \end{aligned} \tag{2.9}$$

Let C_3 be the curve on which $d(V - f(w))/dw = 0$ in (1.41); that is, the curve given by

$$V = \frac{2w(1 - w^2)(1 - \beta w^2)^2}{(1 - \beta w^2)^3 - 4w^2R^{-1}\{1 + (1 - 3\beta)w^2 + \beta w^4\}}. \tag{2.10}$$

Then in the range $0 \leq w < \beta^{-1/2}$, under the condition (2.5), C_3 lies between C_1 and $V = g_1(w)$. This fact is of use in connection with the approach to the limiting form **3.1.**

2.2. The solution curves may be shown to have the following properties:—

(i) The point $(\beta^{-1/2}, 0)$ is a saddle point, the solution curves passing through it being tangent to the two branches of C_2 passing through this point, see **2.1**, (iii).

(ii) For large w there is a family of curves asymptotic to C_1, and another family asymptotic to $V = -\frac{1}{2}Rw$.

(iii) There is an infinite number of solution curves approaching $w = 0$ for large negative V. These may be shown to have the form

$$V = -\tfrac{1}{2}Rw^{-1} + A - (\tfrac{1}{2}R - 2)w + O(w^2), \tag{2.11}$$

where A is an arbitrary constant.

(iv) By a method similar to that used in (**1.3**), (or by substitution of a power series), the asymptotic form of the solution curves through (0, 0), in the neighbourhood of this point, is found to be

$$\begin{aligned} V \sim 2w + \{2(1 - \beta) + 8R^{-1}\}w^3 + \{2(1 - \beta) + 8(5 - 4\beta)R^{-1} \\ + 128R^{-2}\}w^5 + \{2(1 - \beta) + 8(14 - 16\beta + 3\beta^2)R^{-1} \\ + 64(20 - 15\beta)R^{-2} + 3456R^{-3}\}w^7 + \cdots + C'e^{-z} \end{aligned} \tag{2.12}$$

so that by comparison with (2.8) and (2.9), all these solution curves lie above $f(w)$ and $g_1(w)$ for w small enough, as the coefficient of w^7 in (2.12) is greater than the coefficient of w^7 in (2.9) by an amount $192(1 - \beta)R^{-2} + 768R^{-3}$.

The above properties, and a knowledge of the regions of positive and negative curvature and positive and negative slope, enable the solution curves to be sketched, as in

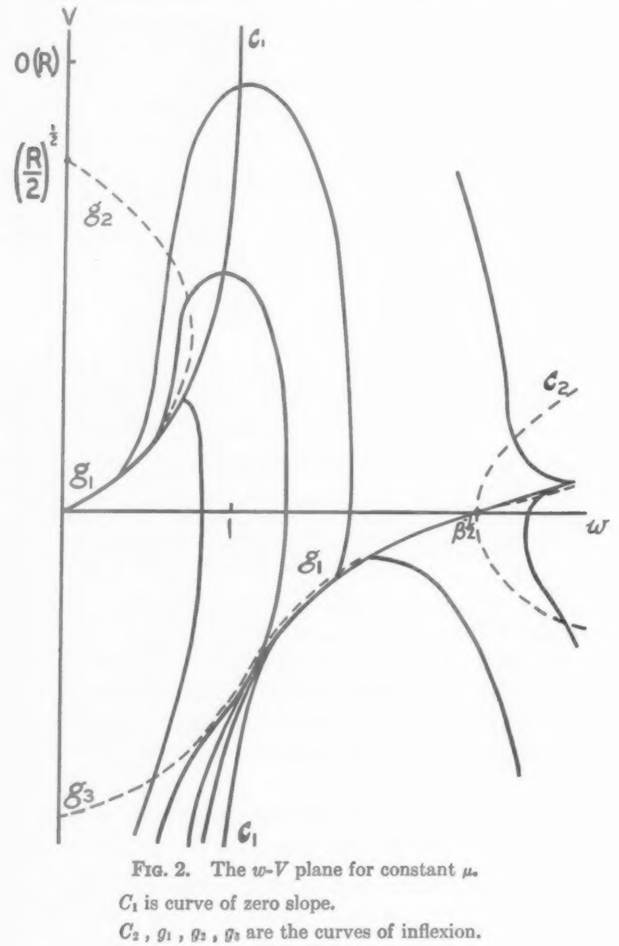

Fig. 2. The w-V plane for constant μ.
C_1 is curve of zero slope.
C_2, g_1, g_2, g_3 are the curves of inflexion.

Fig. 2. It is seen that the solution curves passing through the origin may be divided into three main classes:—

(I) those which pass through $V = g_2(w)$, cross the line $w = \beta^{-1/2}$ and are ultimately asymptotic to C_1, for large w;

(II) Those which pass through $V = g_2(w)$, bend round to cross the w-axis, and are ultimately asymptotic to the negative V-axis (see (2.11));

(III) Those which pass through $V = g_1(w)$, and then cross the w-axis and become asymptotic to the negative V-axis, as for class II. The solutions of the first class, whose curves penetrate into the region of negative temperature are discarded as physically impossible just as in inviscid theory. However, it will be seen later that all possible flows are automatically confined to $0 \leq w < \beta^{-1/2}$ when μ is taken proportional to T.

3. The approach to the limit.

Now that the general shape of the solution curves in the w-V plane has been established, we proceed to investigate whether any limiting curves are approached as $R \to \infty$, and if so, how closely these limiting curves are approximated when R is large. A clue is given by considering the differential equation (1.41) in the form

$$\frac{dV}{dw} = \frac{R}{w^2 V}\left\{\frac{1}{2} V(1 - w^2) - w(1 - \beta w^2)\right\}, \tag{3.1}$$

which shows that, at a fixed point, $dV/dw \to \infty$ as $R \to \infty$. So that considerable portions of the solution curves may be expected to have very large slope. In fact, it will be shown that these portions are very nearly vertical straight lines. The technique used is to find bounding curves which lie on either side of the solution curve considered, and which approach a limiting curve as $R \to \infty$.

3.1. In Fig. 3 consider the solution curve passing through the point $P(b, c)$ on $V = g_2(w)$ so that b and c satisfy equation (2.2) and

$$c = \left\{\frac{R}{2}(1 - b^2)(1 - \beta b^2)\right\}^{1/2}\{1 + o(1)\}, \tag{3.2}$$

but assume that

$$(1 - b^2)^{-1}R^{-1/3} = o(1). \tag{3.3}$$

After a considerable amount of calculation based on the value of the solution derivative on the bounding curves*, the following results may be established.

For $w > b$, the solution curve through P lies above the segment PQ' of the line

$$V = h_\epsilon(w) = c + \frac{R}{b^2}\left\{\frac{1}{2}(1 - b^2) - \frac{b}{c}(1 - \beta b^2)\right\}(1 - \epsilon)(w - b), \qquad 0 < \epsilon < 1, \tag{3.4}$$

where at Q'

$$w = b + \tfrac{1}{2}\epsilon b(1 - b^2), \tag{3.5}$$

and below the tangent

$$V = h(w) = c + \frac{R}{b^2}\left\{\frac{1}{2}(1 - b^2) - \frac{b}{c}(1 - \beta b^2)\right\}(w - b), \tag{3.6}$$

*For example, on PQ' the solution derivative is greater than $h'_\epsilon(w)$ so the solution curve through P cannot cross PQ'.

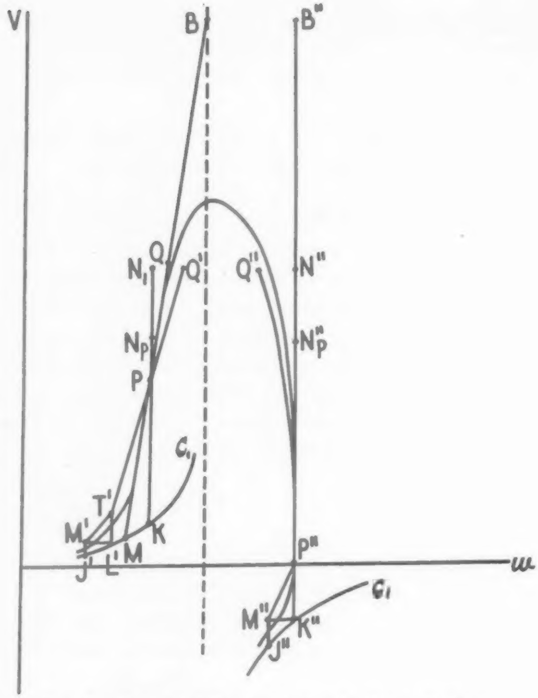

FIG. 3. A typical shock solution in the w-V plane.

for $b \leq w \leq 1$; in particular, it lies below the segment PQ of the tangent, where at Q

$$w = b + \tfrac{1}{2}\,\epsilon(1 - \epsilon)b(1 - b^2). \tag{3.7}$$

For $w < b$, the solution curve through P lies above the segment, PM, of the tangent through P, and below the segment PT' of the line $V = h_\epsilon(w)$, where, at T',

$$w = b_1 = b\left\{1 - \frac{2bc}{R(1 - b^2)(1 - \epsilon)}\,\frac{\epsilon c - 2f(b)}{\epsilon c - \epsilon f(b)}\right\}. \tag{3.8}$$

This solution curve also lies below the straight-line segment $T'M'$, where M' has the coordinates

$$w = b_2 = b_1\{1 - 2b_1(1 - \beta b_1^2)(\lambda - 1)^{1/2}(1 - b_1^2)^{-1/2}R^{-1/2}\}, \qquad V = f(b_1), \tag{3.9}$$

with $\lambda = 2\epsilon^{-1}f(b)/f(b_1)$.

We denote the points on C_1 where $w = b$, $w = b_1$, and $w = b_2$ by K, L' and J' respectively, and as $(b - b_1)$, $(b_1 - b_2)$ and $(b - b_2)$ are all $O(R^{-1/2}(1 - b^2)^{-1/2})$ from (3.8) and (3.9), it follows readily from (2.1) that

$$M'J' = O(R^{-1/2}(1 - b^2)^{-5/2}). \tag{3.10}$$

Also, if $w = b_3$ at the point M, then

$$b - b_3 = O(R^{-1/2}(1 - b^2)^{-1/2}), \tag{3.11}$$

so that when the solution curve through P has approached C_1 to within a distance of $O(R^{-1/2}(1 - b^2)^{-5/2})$ the distance between the solution curve and the vertical line through P is only $O(R^{-1/2}(1 - b^2)^{-1/2})$.

Furthermore, as the solution curve through P cannot cross $V = g_1(w)$ it also cannot cross C_3, (by the remarks following (2.10)), and so, for this solution curve, $v - f(w)$ decreases monotonically to zero as w decreases from b_2 to 0. Thus, for the solution curve through P, the vertical difference between this curve and C_1 is less than $M'J'$ for w between 0 and b_2.

To sum up the progress so far:— if we keep b fixed and let $R \to \infty$, the solution curve through P approaches C_1 for $0 \leq w < b$ (since $M' \to K$), and the vertical line KPN_p, where N_p has the coordinates

$$w = b, \qquad V = c + \frac{R^p\epsilon}{4bc}(1 - b^2)^2(1 - \epsilon)\{c - f(b)\}, \qquad 0 < p < 1. \tag{3.12}$$

When $p = 1$, the distance of the curve from the vertical through P lies between N_1Q and N_1Q', which are both $O(1 - b^2)$, by (3.5) and (3.7).

3.2. Let the solution curve through $w = \beta^{-1/2}$ cross the curve $V = g_2(w)$ at $w = w_i$, then with reference to **2.5**, the curves for which $b > w_i > 0$ belong to class II and those for which $0 < b < w_i$ belong to class I, and by using appropriate bounding curves it may be shown that $1 - w_i = O(1)$. (Actually, it will be shown in **4.2** that $w_i \to \beta^{1/2}$ as $R \to \infty$.) Accordingly, if we suppose that $b > w_i$ for the solution curve already considered, it will cross the w-axis at $w = d$ say, where $0 < d < \beta^{-1/2}$. Furthermore, it may be shown that, if $(1 - b^2)^{-1}R^{-1/3} = o(1)$, then $(d - 1)$ is positive and at least of order $(1 - b)$, and hence $(d - 1)^{-1}R^{-1/3} = o(1)$. This means that b and d differ from unity to at least the same order in R, and lie on opposite sides of unity.

It is seen from (3.1) that a solution curve crosses the w-axis with a vertical tangent, and from Fig. 2, the solution curve always lies to the left of this tangent. For the solution curve through the point P'', $w = d$, $V = 0$, first consider the part with $1 < w \leq d$, $V \geq 0$. This lies to the left of the line $w = d$, and a bounding curve is required which lies to the left of the solution curve. It is possible to find an inclined straight line which has this property, but this is not sufficient, as we need later to take an integral of V^{-1} over the bounding curve, and the result should be finite. We desire a bounding curve which has the same singularity at $w = d$, $V = 0$ as the solution curve, that is, which behaves like $(d - w)^{1/2}$ near this point. The first two terms of the Taylor expansion of V near $w = d$, $V = 0$ give a satisfactory bounding curve, $P''Q''$, given by

$$V = h_1(w) = \{2(1 - \beta d^2)Rd^{-1}\}^{1/2}(d - w)^{1/2} + \frac{d^2 - 1}{3d^2}R(d - w), \tag{3.13}$$

where at Q''

$$w = d\left\{1 - \frac{1}{6}(d^2 - 1)\right\}. \tag{3.14}$$

For the part of the curve $1 < w \leq d$, $V \leq 0$, the solution curve lies to the left of the vertical line segment $P''K''$, and to the right of the line segment $P''M''$, where M'' has the coordinates

$$w = d_1 = d\{1 - 2(1 - \beta d^2)(d^2 - 1)^{-1/2}R^{-1/2}\}, \qquad V = f(d). \tag{3.15}$$

Let the points K'', J'' lie on C_1 with $w = d$ and $w = d_1$ respectively, and let the point N_p'' lie on the line $w = d$, with

$$V = \frac{R^p(d^2 - 1)^2}{18d}\left\{1 + \left[\frac{108d^2(1 - \beta d^2)}{R(d^2 - 1)^3}\right]^{1/2}\right\}, \qquad 0 < p \leq 1, \tag{3.16}$$

so that N_1'' has the same value of V as Q''. Then

$$Q''N_1'' = \frac{1}{6}\,d(d^2 - 1), \tag{3.17}$$

$$K''M'' = \frac{2d(1 - \beta d^2)}{(d^2 - 1)^{1/2}R^{1/2}}, \tag{3.18}$$

and it follows that

$$M''J'' = O(R^{-1/2}(d^2 - 1)^{-5/2}). \tag{3.19}$$

The solution curve, after crossing $M''J''$, then crosses C_2 at $w = d_2$ say, where $1 < d_2 < d_1$, and thereafter lies below it. Thus in the range $1 < w \leq d_2$ the solution curve lies between C_1 and $V = g_1(w)$, and as the difference in height between $V = f(w)$ and $V = g_1(w)$ is $O(R^{-1} \mid w^2 - 1 \mid^{-4}$, (from (2.6)), the solution curves lies within $O(R^{-1/2}(d^2 - 1)^{-5/2})$ of $V = f(w)$ from $w = 1 + O(R^{-1/8}(d^2 - 1)^{5/8})$ to $w = d_1$. Continued backward from this range, the curve crosses the line $w = 1$ with $V < -(1 - \beta)^{2/3}R^{1/3}$ (from 2.3) and is ultimately asymptotic to $w = 0$, as given by (2.11). At the point N_p'', the difference in w between the solution curve and the line $w = d$ is of order $R^{p-1}(d^2 - 1)$, so that as $R \to \infty$, the solution curve approaches C_1, between $1 < w \leq d$, and the straight line $K''P''N_p''$.

In the range $b < w < d$, define the points B, B'' to have respectively the coordinates

$$w = 1, \qquad V = c + \frac{R}{2b^2c}(1 - b^2)(1 - b)\{c - f(b)\}, \tag{3.20}$$

and

$$w = d, \qquad V = c + \frac{R}{2b^2c}(1 - b^2)(1 - b)\{c - f(b)\}. \tag{3.21}$$

Then the solution curve obviously lies above the line $Q'Q''$, and below the lines QB (the continuation of PQ), and BB''. Thus the maximum V attained is of order $(1 - b)^2R$.

The required bounding curves have now been found for the only important class of solution curves, and in the next section, this part of the discussion is completed by finding a relation between b and d, which shows the shock-character of the solution.

4. The "diffuse" shock and the main solution.

It has been shown in the previous section that over some part of the range of w, a typical solution curve of class II approaches the inviscid solution curve in the w-V plane, as $R \to \infty$. It will be shown in this section that the steeply 'humped' part of a typical solution curve of this class approaches a solution appropriate to an ideal shock.

4.1. We require first an estimate of the actual physical distance between the points P and P''. It is sufficient to consider the difference in ξ between these points, and since

$$V = -2dw/d\xi,$$

then

$$\xi_P - \xi_{P''} = 2\int_b^d \frac{dw}{V}. \tag{4.1}$$

From (3.4), (3.5), (3.12), (3.13), and the rest of Sec. 3,

$$\xi_P - \xi_{P''} < 2 \int_b^{w_{Q'}} \frac{dw}{h_\epsilon(w)} + 2 \int_{w_{Q''}}^d \frac{dw}{h_1(w)} + 2(d - b) \min (V_{Q''}^{-1}, V_{Q'}^{-1}), \tag{4.2}$$

and

$$\xi_P - \xi_{P''} > 2 \int_b^1 \frac{dw}{h(w)} + 2(d - 1)[h(1)]^{-1}. \tag{4.3}$$

The integrals which appear in (4.2) and (4.3) are respectively

$$\frac{2b^2c}{R(1 - b^2)(1 - \epsilon)(c - f(b))} \log \left\{1 + \frac{1}{4} b^{-1}R\epsilon(1 - \epsilon)(1 - b^2)^2(c - f(b))\right\}, \tag{4.4}$$

$$\frac{6d^2}{R(d^2 - 1)} \log \left\{1 + \frac{1}{6} d^{-1}R^{1/2}(d^2 - 1)^{3/2}[3(1 - \beta d^2)]^{-1/2}\right\}, \tag{4.5}$$

and

$$\frac{2b^2c}{R(1 - b^2)(c - f(b))} \log \left\{1 + \frac{1}{2} b^{-2}R(1 - b)(1 - b^2)(c - f(b))\right\}, \tag{4.6}$$

so that, for large R, (by what has gone before) they are all

$$O\left\{\frac{\log [R(1 - b)^3]}{R(1 - b)}\right\}. \tag{4.7}$$

Also, by their definitions, $V_{Q''}$, $V_{Q'}$ and $h(1)$ are all $O\{R(1 - b)^2\}$, so that the terms not involving integrals in (4.2) and (4.3) are all

$$O\{R^{-1}(1 - b)^{-1}\}. \tag{4.8}$$

Hence

$$\xi_P - \xi_{P''} = O\left\{\frac{\log [R(1 - b)^3]}{R(1 - b)}\right\} \tag{4.9}$$

for large R.

4.2. With the aid of the above result we are now able to find the asymptotic relation between b and d, which in the limit $R = \infty$, will be seen to be equivalent to Prandtl's relation between the speeds on opposite sides of an ideal shock.

Equation (1.15), (with constant μ), may be put into the form

$$\frac{d}{d\xi}\left\{(1 + \beta)w + \frac{\theta}{w}\right\} = \frac{\theta}{w} + \frac{4}{R}\left\{\frac{d^2w}{d\xi^2} - w\right\}, \tag{4.10}$$

and so

$$\int_{\xi_P}^{\xi_{P''}} \frac{d}{d\xi}\left\{(1 + \beta)w + \frac{\theta}{w}\right\} d\xi = -\frac{2}{R}(V_{P''} - V_P) + \int_{\xi_P}^{\xi_{P''}} \left(\frac{\theta}{w} - \frac{4w}{R}\right) d\xi$$

$$= \frac{2c}{R} - O(\xi_P - \xi_{P''})$$

as θ/w is $O(1)$ between ξ_P and ξ_P''. From (3.2), the first term on the right-hand side is asymptotically

$$\{2R^{-1}(1 - b^2)(1 - \beta b^2)\}^{1/2},$$

while from (4.9), the second term is of higher order, so that

$$\left[(1+\beta)w + \frac{\theta}{w}\right]_{w=b}^{w=d} = \{2R^{-1}(1-b^2)(1-\beta b^2)\}^{1/2}\{1+o(1)\}, \tag{4.11}$$

and as $(d-b)$ is $O(1-b)$,

$$bd = 1 + O(R^{-1/2}(1-b)^{-1/2}). \tag{4.12}$$

In terms of actual speeds, this relation becomes

$$u_1u_2 = a^{*2}\{1 + O(R^{-1/2}(a^* - u_1))^{-1/2}\}, \tag{4.13}$$

where

$$a^* = \left(\frac{2}{\gamma+1}\right)^{1/2} a_0 , \tag{4.14}$$

and as $R \to \infty$, this becomes the Prandtl relation.

With reference to the beginning of **3.2**, where the solution curves belonging to class II were defined as those for which $b > w_i > 0$, we see from (4.12) that as $d \to \beta^{-1/2}$, $b \to \beta^{1/2}\{1 + O(R^{-1/2}(1-b)^{-1/2})\}$, and hence we have

$$w_i = \beta^{1/2}\{1 + O(R^{-1/2}(1-b)^{-1/2})\}. \tag{4.15}$$

4.3. Summary of main solution. From sec. 3 we see that in the w-V plane, a typical solution curve of class II which passes through $P(w_1, V_1)$ on the curve $V = g_2(w)$, approaches the inviscid solution curve in the ranges $0 \le w < w_1$ and $1 < w < w_2$. For large R and fixed w_1 we may summarise the behaviour as follows:—

(i) In $0 \le w \le w_1 - \epsilon_1$, $V \ge 0$, where $\epsilon_1 = O(R^{-1/2})$, V goes from 0 to $f(w) - \eta_1$, $\eta_1 = O(R^{-1/2})$, and $V - f(w)$ is at most $O(R^{-1/2})$.

(ii) In $1 + \epsilon_3 \le w \le w_2 - \epsilon_2$, $V < 0$, $(w_2 = w_1^{-1}[1 + O(R^{-1/2})])$, where $\epsilon_2 = O(R^{-1/2})$, $\epsilon_3 = O(R^{-1/8})$, V goes from $f(1+\epsilon_3) + \eta_3$ to $f(w_2) - \eta_2$, where η_1, $\eta_2 = O(R^{-1/2})$, and $V - f(w)$ is at most $O(R^{-1/2})$.

(iii) (a) In $w_1 - \epsilon_1 \le w \le w + \epsilon_1^{(p)}$, $V > 0$, where $\epsilon_1^{(p)}$ is $O(R^{-p})$, V goes from $f(w_1) - \eta_1$ to $O(R^{1-p})$.

(b) In $w_2 - \epsilon_2^{(p)} \le w \le w_2$, $V > 0$, where $\epsilon_2^{(p)}$ $O(R^{-p})$, V goes from $O(R^{1-p})$ to 0.

(c) In $w_1 + \epsilon_1^{(p)} \le w \le w_2 - \epsilon_2^{(p)}$, $V > 0$, V is $O(R^{-p})$, being $O(R)$ when $p = 0$, that is, when both $w - w_1$ and $w_2 - w$ are $O(1)$.

(d) In $w_2 \ge w \ge w_2 - \epsilon_2$, $V < 0$, V goes from 0 to $f(w_2) - \eta_2$.

(iv) In $1 + \epsilon_3 \ge w \ge 0$, $V < 0$, the curve crosses $w = 1$, with $|V| > (1-\beta)^{2/3}R^{1/3}$, and becomes asymptotic to $w = 0$, as given by (2.11).

The part of the solution described in (iii) above approaches an 'ideal' shock solution as $R \to \infty$, for the equation for the conservation of mass, equation (1.39) and equation (4.12) give, in the limit,

$$\rho_1u_1 = \rho_2u_2 , \tag{4.16}$$

$$\frac{\gamma p}{\rho} = a_0^2 - \frac{\gamma - 1}{2} u^2, \tag{4.17}$$

and

$$u_1u_2 = \frac{2}{\gamma+1} a_0^2 , \tag{4.18}$$

(where the fact that $\xi_P - \xi_{P''} \to 0$ as $R \to \infty$ is used in (4.16)), and these imply the Hugoniot relations; the curve approached is $w = w_1$, $f(w_1) \leq V < \infty$ and $w = w_2$, $f(w_2) \leq V < \infty$.

The actual shock, (as opposed to the limiting 'ideal' shock) has a maximum velocity gradient of order R, which occurs where the solution curve crosses C_1 (within $O(R^{-1})$ of $w = 1$) and its 'thickness' is of order $R^{-1}(1 - w_1)^{-1} \log [R(1 - w_1)^3]$, from (4.9). The 'thickness' has been defined as the distance in the physical plane between P and P'' in Fig. 3. Of course, there is some degree of choice as to the definition of the shock thickness, and this leads to an apparent discrepancy between the above results and the usual result for a plane shock [3], which is that the shock thickness is of order $\mu\rho^{-1}(1 - w_1)^{-1}((1 - w_1)$ is proportional to the shock strength.) The difficulty is resolved if an arbitrary length is introduced in the plane-shock treatment so as to allow the definition of a Reynolds number. Then, for a given shock strength, it is seen that the thickness is proportional to R^{-1} (implied above) only if the velocity gradients at the points measured from are $O(R)$, while if the edges of the shock are taken to be at points where the velocity gradient has fallen to less than $O(R)$, as in the treatment here, the results agree. In practical cases, however, the difference is slight.

From (1.21) and (4.16) to (4.18), the solution approached in the physical plane is given by

$$r = \frac{\kappa}{a_0\rho_0(1 - \beta)^{1/2}} \cdot w^{-1}(1 - \beta w^2)^{(\beta-1)/2\beta}, \qquad 0 \leq w \leq w_1, \tag{4.19}$$

$$r = \frac{\kappa}{a_0\rho_0'(1 - \beta)^{1/2}} \cdot w^{-1}(1 - \beta w^2)^{(\beta-1)/2\beta}, \qquad 1 < w \leq w_1^{-1}, \tag{4.20}$$

where

$$\frac{\rho_0'}{\rho_0} = w_1^2\left\{\frac{1 - \beta w_1^2}{1 - w_1^{-2}}\right\}^{(1-\beta)/(2\beta)}, \tag{4.21}$$

and the suffix $_0$ refers to conditions at the stagnation point at $r = \infty$ (ρ_0 and ρ_0' are different because of the change in entropy through the shock.) From (4.21) we see that a change of w_1 corresponds to a change of the boundary conditions at a given point of the supersonic region, and it is this that determines which of the infinite number of solution curves of class II is selected for given boundary values. This corresponds to a diffuser with constant outlet conditions.

The remainder of the solution curve in the w-V plane, for $0 \leq w \leq 1$, with V negative, approaches in the physical plane the flow starting from a stagnation-point at

$$r = r_s = \frac{\kappa}{a_0\rho_0'}(1 - \beta)^{-1/(2\beta)} \tag{4.22}$$

with infinite pressure and density, and proceeding with infinite velocity gradient up to $w = 1$, where it joins on to the flow given by (4.20) (see Fig. 4). The equations would break down before this second stagnation-point is reached; nevertheless, this theory does not exclude a transition from subsonic flow to supersonic flow without a contraction as, in fact, there is heat addition at the second stagnation point. However, the velocity gradient in this range is so great, that it is difficult to see how this kind of boundary condition could be realised in practice.

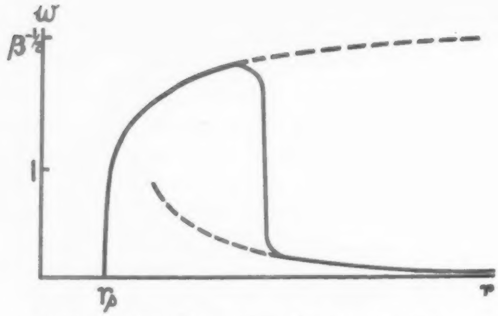

FIG. 4. Typical w-r curve for a viscous gas.

4.4. Some numerical results. This theory could be applied to the flow in the divergent part of a supersonic-subsonic nozzle, neglecting the boundary layer effects. It may be useful for low-density tunnels, as it should be more precise than plane-shock theory. To give some idea of the magnitudes involved, some rough results calculated from **4.1** and **4.3** are given below, in which δ_1 and δ_2 are respectively upper and lower bounds for $(r_1 - r_2)$. The estimates of $(r_1 - r_2)$ could be improved, but not without a considerable amount of labour. The thicknesses involved are of the expected order of magnitude.

w_1	w_2	r_1 cm	T_0 ^0A	$\frac{\rho_1 - \rho_2}{\rho_2}$	ρ_0 gm.cm.$^{-3}$	R	δ_1 micron	δ_2 micron
0.7	1.43	17.98	288	1.04	1.226×10^{-3}	6.02×10^6	6.12	0.93
0.7	1.43	17.98	288	1.04	1.226×10^{-4}	6.02×10^5	48.7	8.0
0.7	1.43	17.98	288	1.04	1.226×10^{-5}	6.02×10^4	402	67
0.9	1.11	15.86	288	0.234	1.226×10^{-3}	6.02×10^6	14.4	2.57
0.9	1.11	15.86	288	0.234	1.226×10^{-4}	6.02×10^5	116	20.5
0.9	1.11	15.86	288	0.234	1.226×10^{-5}	6.02×10^4	899	157

4.5. It may be shown by arguments similar to those employed above that if $P(w_1, V_1)$ lies on $V = g_1(w)$, and $(1 - w_1^2)^{-1}R^{-1/3} = o(1)$, that is, for the curves of class III, the solution curves cross the w-axis before $w = 1$ and then approach $w = 0$ as described before. As $R \to \infty$, the solution curve approaches C_1 in $0 \leq w < w_1$ $(V < 0)$, and the remainder of the curve gives rise to a sort of 'negative shock', similar to that for the curves of class II in $0 < w < 1$, $(V < 0)$.

When $(1 - w_1^2)$ is of order $R^{-1/3}$, (the curves previously excluded) the situation is confused, since there is a transition from curves of the 'shock' type to those of the 'negative shock' type just described as P moves along $V = g_2(w)$ to (w_m, V_m) and then back along $V = g_1(w)$. It is impossible to tell without detailed calculation, possibly involving numerical integration, just where this transition takes place.

It should be noted that none of the solutions obtained has an infinite discontinuity of velocity gradient, even in the limit $R = \infty$, and the flow patterns are quite different from those obtained for the inviscid fluid.

5. The entropy variation.

In this section we will consider the variation of specific entropy over a typical solution curve of the shock type, as an interesting fact arises which, it is believed, was not hitherto known. It is found that there is an entropy maximum within the shock, and in the limit, as the shock becomes infinitely thin, this maximum does not disappear. Of course, the same phenomenon occurs with the plane shock, as is demonstrated.

We define S to be the specific entropy, and start from the well-known equation

$$T\frac{DS}{Dt} = \frac{1}{\rho}\{\Phi + \lambda\nabla^2 T\} \tag{5.1}$$

where Φ is the viscous dissipation function.

For this problem, the equation becomes

$$T\rho u\frac{dS}{dr} = \frac{4}{3}\mu\left\{\frac{u^2}{r^2} - \frac{u}{r}\frac{du}{dr} + \left(\frac{du}{dr}\right)^2\right\} + \lambda\left\{\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right\}, \tag{5.2}$$

and in terms of the non-dimensional quantities from (1.12), (1.13), (1.14), and (1.22),

$$\frac{R\theta}{\mathrm{R}}\frac{dS}{d\xi} = 4\left\{w^2 + \frac{1}{2}wV + \frac{1}{4}V^2\right\} + \frac{3V}{8\beta\sigma}\left\{V\frac{d^2\theta}{dw^2} + \frac{dV}{dw}\frac{d\theta}{dw}\right\}. \tag{5.3}$$

When $\sigma = \sigma_0$, (5.3) becomes

$$\frac{\theta}{\mathrm{R}}\frac{dS}{d\xi} = -\frac{1}{w}\left\{\frac{1}{2}V\left[1 - \left(1 + \frac{4}{R(1+\beta)}\right)w^2\right] - w\left[1 - \beta\left(1 + \frac{4}{R(1+\beta)}\right)w^2\right]\right\} \tag{5.4}$$

$$\doteqdot -\frac{wV}{R}\frac{dV}{dw}. \tag{5.5}$$

The use of (5.5) instead of (5.4) is justified in the same way as the basic approximation used in **1.4**, as differences in w of order R^{-1} only are neglected.

In Fig. 3 we see, then, that $dS/d\xi$ is very small over the regions where the solution curve lies close to the inviscid curve, becomes of order 1 in the interior of the shock and vanishes very close to the point of maximum velocity gradient. In fact, if w_V and w_S denote respectively the values of w at which the velocity gradient and entropy have their maxima, then it may be shown that

$$w_V = 1 - O(R^{-1}(1 - w_1)^{-2}), \tag{5.6}$$

$$w_S = w_V - O(R^{-1}(1 - w_1)^{-2}) \tag{5.7}$$

$$= 1 - O(R^{-1}(1 - w_1)^{-2}). \tag{5.8}$$

By examining the signs of V and dV/dw, it is seen that $dS/d\xi < 0$ in $0 \leq w < w_S$, $V > 0$, and $dS/d\xi > 0$ over the remainder of the curve. This seemingly paradoxical result of decreasing entropy is explained by the fact that as the heat conductivity is not zero, fluid elements are no longer isolated systems. In the range $0 \leq w < w_S$, $V > 0$ that is, on the subsonic 'side' of the shock, heat is continually being conducted backward in the sense of decreasing r, but a given fluid element in this range gives out more heat at the rear than it takes in from the front, and so has a net loss of heat energy.

For a perfect gas, with the neglect of a constant,

$$S = C_v \log(p/\rho^\gamma), \tag{5.9}$$

which becomes, in our notation,

$$S - S_0 = C_v \log \{\theta(wr)^{2\beta/(1-\beta)}\}, \tag{5.10}$$

where

$$S_0 = \frac{C_v}{1 - \beta} \log \{RT_0\kappa^{-2\beta}(1 + \beta)^\beta\}. \tag{5.11}$$

So, at the second stagnation point, $S = -\infty$, and with this additional information, the sketch curve of entropy variation with ξ (Fig. 5) may be drawn. If we denote by

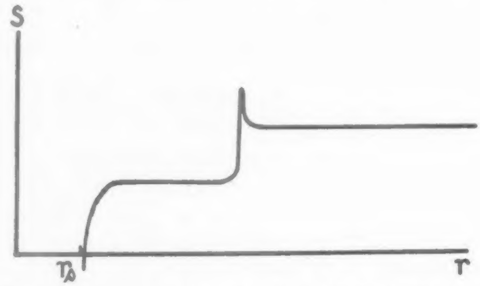

FIG. 5. Entropy variation for a typical shock solution.

S_1, S_2 and S_{max} respectively, the entropy when $w = w_1$, w_2 and w_S, we have from (5.10)

$$S_1 - S_0 = C_v \log \{\theta_1(w_1r_1)^{2\beta/(1-\beta)}\},$$

$$S_2 - S_0 = C_v \log \{\theta_2(w_2r_2)^{2\beta/(1-\beta)}\},$$

and

$$S_{max} - S_0 = C_v \log \{\theta_S(w_Sr_S)^{2\beta/(1-\beta)}\}.$$

In the limit, as $R \to \infty$, $w_2 \to w_1^{-1}$, $w_S \to 1$ from (5.9), and r_1, r_2 and r_S tend to a common value, so that

$$S_{max} - S_1 = C_v \log \left\{\frac{1 - \beta}{1 - \beta w_1^2} w_1^{2\beta/(\beta-1)}\right\}, \tag{5.12}$$

$$S_1 - S_2 = C_v \log \left\{\frac{1 - \beta w_1^2}{w_1^2 - \beta} w_1^{2(1+\beta)/(1-\beta)}\right\}, \tag{5.13}$$

the last being the usual result for an ideal shock.

For plane flow, with constant viscosity, we find the solution

$$\frac{(w - w_1)^{w_1}}{(w_2 - w)^{w_2}} = D \exp\left\{-\frac{3}{4}\frac{m}{\mu}\frac{(w_2 - w_1)}{1 + \beta}x\right\}, \tag{5.14}$$

where $m = \rho u$ and $\sigma = 3/4$, [4], which is seen to give a shock-type flow, the 'speeds' w_1, w_2 being attained at $x = -\infty$ and $x = \infty$ respectively. In this case

$$\frac{\theta}{R}\frac{dS}{dx} = -\frac{3m}{4\mu(1 + \beta)} \cdot \frac{1 - w^2}{w^2}(w_2 - w)(w - w_1). \tag{5.15}$$

This clearly exhibits the entropy maximum which now falls exactly at $w = 1$, (the point of inflexion of the w-x curve) while the relations (5.13) and (5.14) for the entropy differences $S_{max} - S_1$, and $S_1 - S_2$ now hold exactly.

6. Viscosity varying with temperature.

In this section we consider briefly the case of μ varying directly with temperature. The results are qualitatively the same for the shock type curves, but the whole flow is now automatically confined to the region of positive temperature.

The equations to be investigated are (1.15) and (1.16) (with $C = 1$ as before). Assume that

$$\mu = \mu_0 \theta, \tag{6.1}$$

and hence

$$R = R_0/\theta, \tag{6.2}$$

where

$$R_0 = \frac{3\kappa}{\mu_0(1 + \beta)}, \qquad \text{a constant.} \tag{6.3}$$

Equation (1.16) now has the form

$$\theta - 1 + \beta\left\{1 + \frac{4\theta}{R_0(1 + \beta)}\right\}w^2 - \frac{\theta}{R_0(1 + \beta)}\left\{\frac{3}{\sigma}\frac{d}{d\xi}(\theta - 1) + 8\beta w \frac{dw}{d\xi}\right\} = 0, \tag{6.4}$$

and to find an integrating factor as in **1.4**, we must take

$$\sigma = \frac{3}{4} \cdot \frac{R_0(1 + \beta) + 4\theta}{R_0(1 + \beta) + 4\beta w^2}, \tag{6.5}$$

which is not constant. However, for the range of w in question the departure of σ from the constant value 3/4 is so small that it may be neglected for practical purposes. With this value of σ, if we again define

$$\begin{aligned} E &= \theta - 1 + \beta\left\{1 + \frac{4}{R(1 + \beta)}\right\}w^2 \\ &= \theta - 1 + \beta\left\{1 + \frac{4\theta}{R_0(1 + \beta)}\right\}w^2, \end{aligned} \tag{6.6}$$

then equation (6.4) has the formal solution

$$E = A \exp\left[\int^{\xi} \frac{R_0(1 + \beta)}{4\theta}\left\{1 + \frac{4\theta}{R_0(1 + \beta)}\right\} d\xi\right]. \tag{6.7}$$

Physically, θ must be bounded near the stagnation point at ∞, and if we assume that it is integrable in this neighbourhood, then A must be zero for E to remain finite as $\xi \to \infty$, and so, as in **1.4**, $E \equiv 0$, and hence

$$\theta = R_0(1 + \beta)\frac{1 - \beta w^2}{R_0(1 + \beta) + 4\beta w^2}\cdot \tag{6.8}$$

With change of independent variable to w, and neglect of terms of order R_0^{-1} compared with 1, as in **1.4**, eq. (1.16) takes the form

$$\frac{w^2 V}{R_0}(1 - \beta w^2)\frac{dV}{dw} = \frac{1}{2} V(1 - w^2) - w(1 - \beta w^2) + 2\beta R_0^{-1} w^3 V^2 \tag{6.9}$$

and, with

$$Z = V\theta, \tag{6.10}$$

$$\frac{w^2 Z}{R_0}\frac{dZ}{dw} = \frac{1}{2} Z(1 - w^2) - w(1 - \beta w^2)^2. \tag{6.11}$$

The solution curves for (6.11) are sketched in Fig. 6, and it is seen that in the range

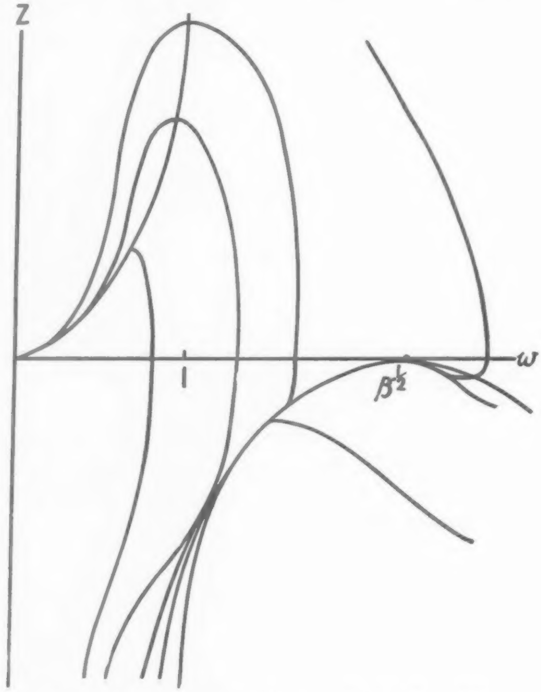

FIG. 6. Solution curves for $\mu \propto T$ with $Z = VO$.

$0 \leq w < \beta^{-1/2}$, their behaviour is very similar to those discussed before in Fig. 3. In fact, the resemblance between (6.11) and (1.41) is striking, and exactly the same technique may be used to discuss this equation. It is obvious that for the 'shock' type curves, similar results will be obtained for Z as were obtained previously for V, and as the variation of θ is only of order 1 for the range $0 \leq w \leq \beta^{-1/2}$, we can carry over the same qualitative results for V for this new equation. (Only coefficients will be affected, orders of magnitude will be unaltered.) It is only near $w = \beta^{-1/2}$ that the results are different, as θ vanishes here. To investigate the behaviour more closely, we may now turn to the $(w\text{-}V)$ plane.

From (6.9), dV/dw is zero when

$$\frac{1}{2} V(1 - w^2) - w(1 - \beta w^2) + \frac{2\beta w^3 V^2}{R_0} = 0, \tag{6.12}$$

provided that neither $V = 0$, $w = 0$ nor $w = \beta^{-1/2}$. Equation (6.12) has the two solutions

$$V_+ = \frac{R_0}{4\beta w^3}\left\{-\frac{1}{2}(1 - w^2) + \left[\frac{1}{4}(1 - w^2)^2 + \frac{8\beta w^4}{R_0}(1 - \beta w^2)\right]^{1/2}\right\}, \tag{6.13}$$

and

$$V_- = \frac{R_0}{4\beta w^3}\left\{-\frac{1}{2}(1 - w^2) - \left[\frac{1}{4}(1 - w^2)^2 + \frac{8\beta w^4}{R_0}(1 - \beta w^2)\right]^{1/2}\right\}, \tag{6.14}$$

and provided that $(1 - w^2)^2 R_0 \gg 1$, that is $| 1 - w |^{-1} R_0^{-1/2} = o(1)$, we have

$$\begin{aligned} V_+ &= f(w)(1 + o(1)), \qquad 0 \leq w < 1, \\ &= \frac{(w^2 - 1)R_0}{4\beta w^3}(1 + o(1)), \qquad 1 < w \leq \beta^{-1/2}, \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} V_- &= \frac{-(1 - w^2)R_0}{4\beta w^3}(1 + o(1)), \qquad 0 \leq w < 1, \\ &= f(w)(1 + o(1)), \qquad 1 < w \leq \beta^{-1/2}. \end{aligned} \tag{6.16}$$

Thus V_+ in $0 \leq w < 1$, and V_- in $1 < w \leq \beta^{-1/2}$, lie very close to the curve C_1 which occurred before, with a small neighbourhood of $w = 1$ of order $R^{-1/2}$ excluded (this small neighbourhood lies inside the neighbourhood of order $R^{-1/3}$ previously excluded for the important curves). There is no longer an infinity at $w = 1$, and V_+ and V_- cross $w = 1$ at a height of order $R^{1/2}$.

For large V of order R,

$$\frac{w^2 V}{R_0}\frac{d^2 V}{dw^2} = \frac{2\beta w V^2}{(1 - \beta w^2)^2}\left\{1 - w^2 + \frac{4\beta w^3}{R_0} V + o(1)\right\}, \tag{6.17}$$

and we see that in $1 < w < \beta^{-1/2}$, $d^2V/dw^2 > 0$ above V_+ and $d^2V/dw^2 < 0$ below V_+, so that we may now proceed to sketch the solution curves, as in Fig. 7. The solution curves which do not bend round to cross the w axis, now go off to $V = +\infty$ as $w \to \beta^{-1/2} - 0$, and in fact, all the possible flows starting with $0 \leq w < \beta^{-1/2}$ are confined to this range; the line $w = \beta^{-1/2}$ has become a barrier. The speed at which the acceleration is a maximum is now only within $O(1)$ of $w = 1$, and lies between $w = 1$ and $w = \beta^{-1/2}$, but the entropy maximum still lies very close to the point of maximum acceleration.

When $w_1 < \beta^{-1/2}$ (in the previous notation), an incomplete shock is formed, starting from $w = \beta^{-1/2}(\theta = 0)$ with infinite acceleration. This again leads to an impossible boundary condition as regards the production of this flow in practice. Apart from this, the flow patterns are fundamentally the same as those for constant viscosity.

The case of the vortex source may be discussed in an approximate manner with similar results to those already obtained.

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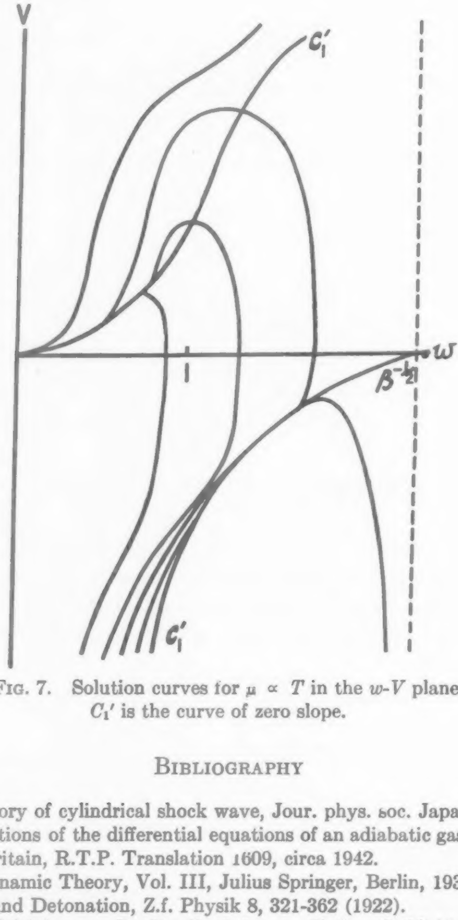

FIG. 7. Solution curves for $\mu \propto T$ in the w-V plane.
C_1' is the curve of zero slope.

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APPROXIMATE ANALYSIS OF STRUCTURES IN THE PRESENCE OF MODERATELY LARGE CREEP DEFORMATIONS*

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1. Introduction. When creep strains of the order of magnitude of one or two percent develop during the lifetime of a structure, it is often permissible to disregard the primary phase of the creep deformations and to base the analysis solely on the secondary or steady phase of creep. For the metals used in structures the experimentally established secondary creep law is generally given in the form

$$e_{11} = K\sigma_{11}^{n}\,, \tag{1}$$

where e_{11} is the tensile strain rate caused by uniaxial tension in direction 1, σ_{11} is the corresponding tensile stress, and K and n are constants. When the creep strains are large (of the order of magnitude of 0.01), the elastic deformations can often be neglected in the calculations, as will be demonstrated by means of an example. Thus the limiting state of stress and strain approached as the creep strain becomes large as compared to the elastic strain can be determined on the basis of a simple non-linear stress-strain rate law.

It is believed that structural analyses based on the assumptions stated are satisfactory for supersonic guided missiles whose surface is heated to high temperatures by the air flow. As guided missiles are generally used only for a single flight and not over long periods of time like piloted airplanes, their structure can be permitted to undergo large permanent deformations.

2. The elastic analogue. It will now be shown that the stress distribution in a body whose deformations are governed by a generalized version of the non-linear creep law of Eq. 1 is the same as that in a non-linear perfectly elastic body provided the elastic stress-strain law and the boundary conditions are suitably chosen. Following Prager's suggestions for the representation of the stress-strain laws of strain-hardening materials [1]†, the uniaxial stress-strain rate law of Eq. 1 is generalized to read

$$E = f(J_2\,,\,J_3)[p(J_2\,,\,J_3)T + q(J_2\,,\,J_3)S'], \tag{2}$$

where

$$T = S'^2 - (2/3)J_2I, \tag{2a}$$

E is the strain rate tensor, S' the stress deviation tensor, and I the identity tensor. The typical component of the stress deviation tensor is defined as

$$s'_{ij} = s_{ij} - (1/3)s_{kk}\delta_{ij}\,, \tag{2b}$$

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†Numbers in brackets refer to the References.

and s_{ij} is the typical component of the stress tensor. The first invariant of the stress deviation naturally vanishes:

$$J_1 = s'_{ii} = 0. \tag{3a}$$

The second and third invariants are defined as

$$J_2 = (1/2)s'_{ij}s'_{ij} , \tag{3b}$$

$$J_3 = (1/3)s'_{ij}s'_{jk}s'_{ki} . \tag{3c}$$

In Eq. 2 the symbols $p(J_2 , J_3)$ and $q(J_2 , J_3)$ denote polynomial functions, and $f(J_2 , J_3)$ an arbitrary function of J_2 and J_3 . These functions must be determined from empirical data to be obtained from creep tests. Eq. 2 is meant for use only when the strain is small as compared to unity (say 0.01); under such conditions it implies that the creep deformations are inextensional ($e_{ii} = 0$).

If a body which follows this stress-strain rate law is subjected to given body forces $\varphi_i(x, t)$, (where x is understood to represent the three Cartesian coordinates of space) and to given surface tractions $T_i(x, t)$ on a portion S_1 of its surface while the points on the remainder S_2 of the surface are slowly displaced with given velocities $V_i(x, t)$ (so slowly that the resulting inertia forces are small as compared to the forces corresponding to the stresses S and surface tractions T), at a generic instant t the stress field $s_{ij}(x, t)$ and the velocity field $v_i(x, t)$ throughout the body must satisfy the following equations:

$$(\partial s_{ij}/\partial x_j) + \varphi_i = 0, \tag{4}$$

$$(\partial v_i/\partial x_j) + (\partial v_j/\partial x_i) = 2f(J_2 , J_3)[p(J_2 , J_3)t_{ij} + q(J_2 , J_3)s'_{ij}]. \tag{5}$$

The three equilibrium equations (4) and the six stress-strain rate equations (5), together with the boundary conditions

$$s_{ij}n_j = T_i \qquad \text{on} \qquad S_1 , \tag{6a}$$

$$v_i = V_i \qquad \text{on} \qquad S_2 , \tag{6b}$$

define the stress and velocity fields in the body.

The analoguous perfectly elastic body is required to follow the stress-strain law

$$E^* = f(J_2 , J_3)[p(J_2 , J_3)T + q(J_2 , J_3)S'] \tag{7}$$

in which the only new symbol, E^*, represents the strain tensor whose typical component is ϵ_{ij} . The body forces and the surface tractions are kept unchanged but the velocities $V_i(x, t)$ prescribed on surface S_2 are replaced by displacements $U_i(x, t)$ equal to $V_i(x, t)$ in magnitude and direction. Under these conditions the displacement field $u_i(x, t)$ and the stress field $s_{ij}(x, t)$ in the elastic body are defined by Eqs. 4 to 6 if $v_i(x, t)$ is replaced by $u_i(x, t)$ and $V_i(x, t)$ by $U_i(x, t)$. Consequently to any solution of the elastic problem there corresponds a solution of the creep problem and the stress distribution is the same in the two solutions.

Similar results were published earlier for linear visco-elastic materials by Alfrey [2] and Tsien [3].

3. Pin-jointed truss. As an example for the use of the analogue the stresses will now be calculated in the bars of the pin-jointed framework shown in Fig. 1. One end of each

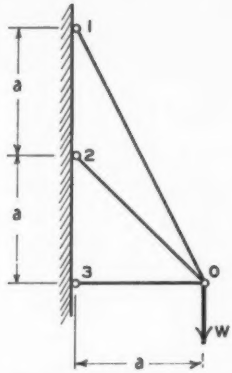

FIG. 1. Pin-Jointed Framework

bar is attached to a rigid wall. Bar 0–3, which is in compression when the load W is applied to joint O, is assumed to be perfectly rigid and braced in the direction perpendicular to the plane of the truss so as to prevent its buckling. The material of the other two bars is subject to creep in accordance with the law

$$e = (\sigma/\lambda)^n, \tag{8}$$

where e is the rate of creep in uniaxial tension, σ the tensile stress, and λ and n are constants. Because of the analogue the creep problem can be replaced by a problem in non-linear elasticity with the stress-strain law.

$$\epsilon = (\sigma/\lambda)^n, \tag{9}$$

where ϵ is the tensile strain.

It might be mentioned here that the behavior of pin-jointed structures was investigated by Meacham [4] on the assumption of a linear creep law.

3a. Load W prescribed. When the load W is prescribed, the stress distribution can be calculated from the requirement that the complementary energy stored in the bars must be a minimum. If the force in bar 0–2 is designated as X, it follows from the conditions of equilibrium that the forces in the three bars are

$$F_{0-1} = (\sqrt{5}/2)W - (\sqrt{10}/4)X, \quad F_{0-2} = X, \quad F_{0-3} = -(1/2)W - (\sqrt{2}/4)X \tag{10}$$

As the complementary energy per unit volume (U'/V) is defined as

$$(U'/V) = \int_0^\sigma \epsilon \, d\sigma, \tag{11}$$

substitution from Eq. 9 and integration yield

$$(U'/V) = [\lambda/(n+1)](\sigma/\lambda)^{n+1}. \tag{12}$$

With A designating the common cross-sectional area of the two elastic bars, the strain energy stored in the system becomes

$$U' = [(n+1)\sigma\lambda^n A^n]^{-1}\{\sqrt{5}[(\sqrt{5}/2)W - (\sqrt{10}/4)X]^{n+1} + \sqrt{2}\, X^{n+1}\}. \tag{13}$$

In accordance with the complementary energy principle the differential coefficient of U' with respect to X must vanish. After some manipulations this requirement can be written in the form

$$(X/W) = (\sqrt{5}/2)[(4/5)^{1/n} + (\sqrt{10}/4)]^{-1}. \tag{14}$$

After (X/W) is computed from Eq. 14 for any given n, the values of F_{0-1} and F_{0-2} can be calculated from Eqs. 10. The results of such computations are plotted in Fig. 2.

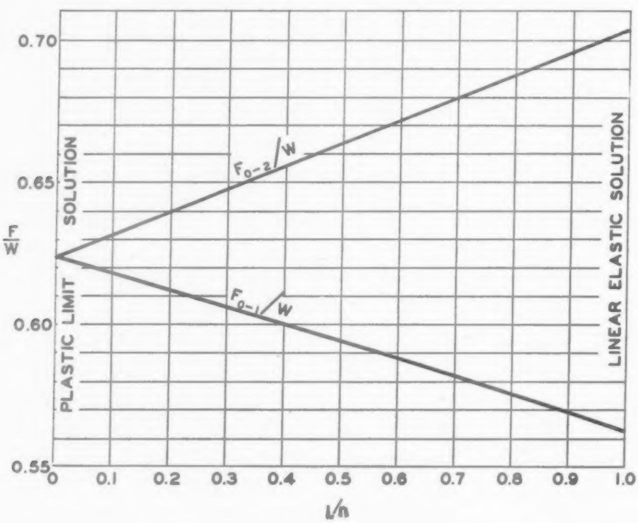

FIG. 2. Forces in Bars of Pin-Jointed Framework

When $n = 1$, the linear elastic solution is obtained; and when $n \to \infty$, the solution corresponds to the principles of limit analysis.

3b. Velocity V_0 prescribed. If the velocity V_0 of joint 0 is prescribed in the creep problem rather than the load W, in the analogous elastic problem the elastic displacement U_0 of joint 0 must be given. The potential V' of the unknown reaction force R at 0 is

$$-V' = RU_0 \tag{15}$$

with both R and U_0 considered positive when directed downward. The forces in the bars caused by the unknown reaction R can be calculated from Eqs. 10 if W is replaced by R. The complementary energy can be obtained in a similar manner from Eq. 13. The compatibility conditions are in this case

$$\partial(U' + V')/\partial X = 0, \tag{16a}$$

$$\partial(U' + V')/\partial R = 0. \tag{16b}$$

As a consequence of the first of these equations Eq. 14 again holds provided W is replaced by R. It follows from the second equation that

$$(R/A) = (2/\sqrt{5})[(4/5)^{1/n} + (\sqrt{10}/4)]\lambda(U_0/2a)^{1/n}. \tag{17}$$

3c. More accurate solution of the creep problem. The problem just discussed is so simple that it could have been solved on the basis of geometric considerations of the deformations without recourse to energy methods. For the same reason it can also be analyzed in the case when the deformations are governed by a more complex creep law than the one represented by Eq. 8. Such a more complete analysis will now be presented in order to show that the initial (linear) elastic stress distribution in an actual structure is rapidly replaced by a distribution that, for all practical purposes, is identical with the stress distribution derived from the (non-linear) elastic analogue.

The creep law will be assumed as

$$e = (1/E)(d\sigma/dt) + (\sigma/\lambda)^n, \tag{18}$$

where t is time. If e_1 is the strain rate in bar 0–1, the rate of elongation $d\Delta L_1/dt$ in the bar is

$$d\Delta L_1/dt = \sqrt{5}\, ae_1\ . \tag{19a}$$

Similarly in bar 0–2

$$d\Delta L_2/dt = \sqrt{2}\, ae_2\ . \tag{19b}$$

The geometric condition imposed upon these deformations can be derived from Fig. 3.

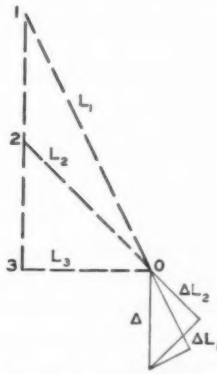

FIG. 3. Displacement of Joint O

When the length of bar 0–1 is increased from L_1 to $L_1 + \Delta L_1$ the length of bar 0–3 remains L_3 because this bar was assumed to be perfectly rigid. The new position of joint 0 can then be found by drawing an arc of a circle from point 1 with a radius $L_1 + \Delta L_1$, a second arc of a circle from point 3 with a radius L_3 , and by determining the point of intersection of the two arcs. When the deformations are small, the arcs of circles can be replaced by straight line segments perpendicular to the original directions of bars 0–1 and 0–3, respectively. This was done in the figure and the vertical displacement Δ of joint 0 was obtained as

$$\Delta = (\sqrt{5}/2)\Delta L_1\ . \tag{20a}$$

The same vertical displacement must result if the geometric construction is carried out for bar 0–2. Hence

$$\Delta = \sqrt{2}\,\Delta L_2\ . \tag{20b}$$

When these two equations are differentiated with respect to time and their right-hand members are equated to each other, the following condition is obtained:

$$(\sqrt{5}/2)\, d\Delta L_1/dt = \sqrt{2}\, d\Delta L_2/dt. \tag{21}$$

Substitutions from Eqs. 18 and 19 yield

$$(5/4)[(1/E)(d\sigma_1/dt) + (\sigma_1/\lambda)^n] = (1/E)(d\sigma_2/dt) + (\sigma_2/\lambda)^n. \tag{22}$$

When the load W is prescribed at joint 0, equilibrium requires that

$$(W/A) = (2/\sqrt{5})\sigma_1 + (1/\sqrt{2})\sigma_2\,. \tag{23}$$

Solution of this equation for σ_2, substitution in Eq. 22, and manipulations result in the differential equation

$$(dx/dt) = K[(1 - bx)^n - (cx)^n], \tag{24}$$

where

$$\begin{gathered} x = (A\,\sigma_1/W), \qquad b = (4/5)^{1/2}, \qquad c = (5/4)^{1/n}(1/2)^{1/2}, \\ K = [2^{n/2}(W/A\lambda)^{n-1}(E/\lambda)]/[(5/4) + (8/5)^{1/2}]. \end{gathered} \tag{25}$$

After separation of the variables and integration the following solution is obtained:

$$t = (1/K)\int_{x_0}^{x} dx/[(1 - bx)^n - (cx)^n]. \tag{26}$$

A numerical example was worked out in which the material of the bars was 52S-H38 aluminum alloy. At 400°F the material constants can be taken as $n = 5$,

$$\lambda = 25{,}000 \text{ hr.}^{1/5} \text{ lb. per sq. in.}$$

on the basis of creep tests carried out by Dorn and Tietz [5]. From Table 3.1211(b) in ANC-5 [6] Young's modulus can be estimated as 9×10^6 lb. per sq. in. With these values one obtains $K = 487$ per hour if the load is assumed to be 22,000 lbs. The load is applied at $t = 0$ and dynamic effects are disregarded. Initially the fully (linearly) elastic solution must prevail; correspondingly $x = 0.56233$ when $t = 0$. The state of fully developed creep is reached only as t approaches infinity; then, in the limit, $x = 0.61206$. Evaluation of the integral in Eq. 26 yielded corresponding values of x and t which were plotted in Fig. 4. It can be seen from the figure that after the lapse of about 70 sec. x is 0.61 which is only one-third of one percent less than the fully developed creep value 0.61206; (the difference between 0.61 and 0.61206 is about four percent of the difference between the elastic and the fully developed creep values). As the highest stress in any bar is about 15,000 lb. per sq. in. (in bar 0–2 at $t = 0$), the maximum creep rate is about 0.0777 per hour. Hence the total creep strain developed in bar 0–2 in the first 70 sec. after the load is applied is less than 0.00151. This should be compared with the maximum elastic strain in the bar which is 0.00167.

Thus the conclusion is reached that in this problem in good approximation the state of fully developed creep is reached at a time when the creep strain is about equal to the elastic strain. The effect of the elastic stresses can therefore be neglected when the state of stress and strain is investigated at moderately large creep deformations (at creep strains of the order of magnitude of 0.01).

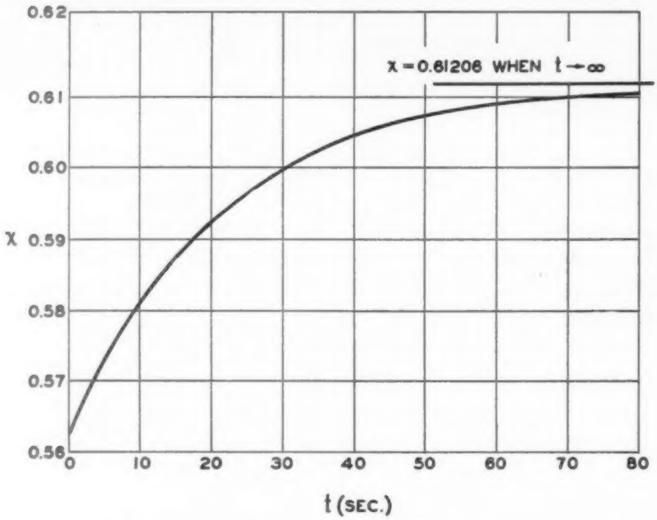

FIG. 4. Variation in Non-dimensional Force in Bar $O - 1$ with Time

Finally it is of interest to check whether the time interval of 70 sec. is short as compared to the time needed for the necking and rupture of the most highly loaded bar. In an earlier paper [7] the author derived the formula

$$t_{cr} = (1/n)/(\sigma_0/\lambda)^n, \tag{27}$$

where t_{cr} is the time needed for rupture and σ_0 is the engineering stress at the beginning of the creep test. It is reasonable to take $\sigma_0 = 14{,}100$ lb. per sq. in. corresponding to the fully developed state of creep in bar 0–2. With this value one obtains $t_{cr} = 3.55$ hours. Thus there is a long period of time during which the fully developed creep solution is valid.

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ORTHOGONAL EDGE POLYNOMIALS IN THE SOLUTION OF BOUNDARY VALUE PROBLEMS*

BY

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1. Introduction. The method presented below for solving boundary value problems is applicable when the problem can be formulated as minimization of some integral. One constructs orthogonal boundary polynomials f_n appropriate to certain boundary conditions, enlarges them by factors g_n—obtained from Euler-Lagrange variational equations—to product functions

$$\varphi_n = g_n f_n \tag{1a}$$

defined over the entire domain, with boundary values

$$\varphi_{nb} = f_n \tag{1b}$$

and expands the prescribed boundary value—say, Φ_b—of the unknown function Φ into the edge polynomials

$$\Phi_b = \sum c_n f_n \, . \tag{2}$$

Then,

$$\Phi \sim \sum c_n \varphi_n = \sum c_n g_n f_n \tag{3}$$

constitutes an approximate solution of the problem. An interesting aspect of the method is that once the general expression for the polynomials f_n has been found, any eigenvalue and its associated eigenfunction can be determined without prior determination of the lower modes.

Another interesting aspect is that while the functions f_n are—by definition—precisely orthogonal, with respect to some suitable weight function ρ,

$$\langle f_n f_m \rangle \equiv \int_{\text{boundary}} \rho \, f_n f_m \, ds = \delta_{nm} \tag{4}$$

the derivatives $\partial f_n/\partial s$, $\partial f_m/\partial s$ (if s denotes the coordinate along the boundary), while not orthogonal, can be regarded—in the problems to be considered—as being approximately orthogonal, in analogy to the precise orthogonality of the derivatives $\partial \Phi_n/\partial s$, $\partial \Phi_m/\partial s$ of the exact eigenfunctions.

In "first approximation" our approach thus neglects the derivative coupling terms altogether and yields the product eigenfunctions (1a) discussed above. In "second approximation" products of $\partial f_n/\partial s$, $\partial f_m/\partial s$ are retained in the variational integral if

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[2]The Knolls Atomic Power Laboratory is operated for the Atomic Energy Commission by the General Electric Co. The work reported here was carried out under contract No. W-31-109 Eng-52. The authors are indebted to Mrs. Jean Born for help with the numerical work.

$m = n$ or $n \pm 1$ (or $m = n$, $n \pm 2$ when due to symmetry conditions $\partial f_n/\partial s$, $\partial f_m/\partial s$ turn out to be precisely orthogonal[3]), and only higher order coupling terms are neglected. In such a case the approximate solution—call it now ψ_n—will appear as a linear combination

$$\psi_n = G_{n,n-1} f_{n-1} + G_{nn} f_n + G_{n,n+1} f_{n+1} \tag{5a}$$

of product eigenfunctions, with boundary value

$$\psi_{nb} = f_n .$$

While the polynomials f_n of the second approximation are the same as those of the first approximation, the G_n functions, again determined from a variational equation, will differ from the g_n functions. Higher approximations are constructed similarly.

Functional developments of type (3) where $f_n(x, y)$ is some assumed function, $g_n(x)$ is a function determined from an Euler variational equation, have been employed previously by Kantorovitch[4] and Poritsky[5]. The novel feature of the present approach is the systematic development of orthonormal sets f_n . The determination of these sets greatly simplifies subsequent calculations as one is allowed (in first approximation) to regard the various modes involved as uncoupled.

The method described above was first used to prove St. Venant's principle for a plane rectangular elastic region subject to selfequilibrating edge tractions.[6] It was found that the problem could be formulated as a biharmonic eigenvalue problem. The present paper illustrates use of the "first approximation" technique for a simpler group of problems, those relating to the harmonic equation. Section 6 discusses the wave equation and illustrates also the method of the "second approximation."

2. The potential is an isosceles triangle. In this section we shall solve the equation

$$\nabla^2 \Phi = 0 \tag{6}$$

in the triangular region shown in Fig. 1, subject to the following boundary conditions.

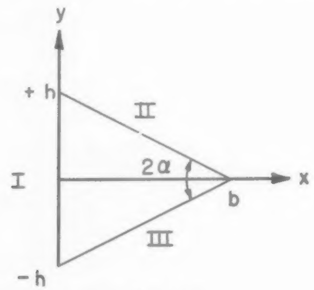

Fig. 1.

[3]As is the case for instance when f_n is an even function, f_{n+1} an odd function.

[4]*Mathematical theory of elasticity*, by I. S. Sokolnikoff, McGraw Hill, 1946, p. 315.

[5]*Reduction of the Solution of Certain Partial Differential Equations to Ordinary Differential Equations*, by H. Poritsky, Proc. Fifth Int. Cong. of Appl. Mech. 1938.

[6]*The end problem of rectangular strips*, by G. Horvay, J.A.M. 1953, p. 87. See also discussion of the paper in the issue of Sept. 1953.

Along the boundaries II, III where

$$y = \pm y_b \equiv \pm h(1 - x/b), \tag{7a}$$

we prescribe

$$\Phi_{II} = \Phi_{III} = 0. \tag{7b}$$

Along the boundary I, where

$$x = 0, \qquad -h \leq y \leq +h, \tag{8a}$$

we prescribe

$$\Phi_I = \text{arbitrary function}. \tag{8b}$$

The integral to be minimized is

$$I = \int_0^b \int_{-y_b}^{+y_b} [(\partial\Phi/\partial x)^2 + (\partial\Phi/\partial y)^2]\, dy\, dx. \tag{9}$$

The lowest degree polynomial $f(y)$ which satisfies

$$f(x, \pm y_b) = 0, \qquad \langle f^2(0, y)\rangle \equiv \int_{-y_b}^{+y_b} f^2(x, y)\, dy\bigg|_{x=0} = \int_{-h}^{+h} f^2(0, y)\, dy = 1 \tag{10}$$

is

$$f_0 = \left(\frac{15}{16h}\right)^{1/2}\left(1 - \frac{y^2}{y_b^2}\right). \tag{11}$$

The higher degree polynomials, obtained by orthonormalization with respect to the lower ones, are

$$\begin{aligned} f_1 &= (105/16h)^{1/2}(y/y_b)[1 - y^2/y_b^2], \\ f_2 &= (3/4)^{1/2}[1 - 7y^2/y_b^2]f_0\,, \\ f_3 &= (11/4)^{1/2}[1 - 3y^2/y_b^2]f_1\,, \\ f_4 &= (91/128)^{1/2}[1 - 18(y/y_b)^2 + 33(y/y_b)^4]f_0\,, \cdots \end{aligned} \tag{12}$$

Introducing the notation

$$F_n = \{F\}_n\,,$$

$$\frac{\partial f(x, y)}{\partial x} = f_x\,, \qquad \frac{\partial f(x, y)}{\partial y} = f_y\,, \qquad \frac{dg}{dx} = g', \qquad \int_{-y_b}^{+y_b} ff_y\, dy = \langle ff_y\rangle \tag{13}$$

and writing

$$\Phi \sim \sum A_n\varphi_n = \sum A_n g_n(x) f_n(x, y), \tag{14}$$

we write Eq. (9) in the form:

$$I = \sum_{n,m} A_n A_m \int_0^b \int_{-y_b}^{+y_b} [\{gf_y\}_n \cdot \{gf_y\}_m + \{gf_x + g'f\}_n \cdot \{gf_x + g'f\}_m]\, dy\, dx. \tag{15}$$

To first approximation this is

$$I = \sum_n A_n^2 \int_0^b \{\langle f^2\rangle g'^2 + 2\langle ff_x\rangle gg' + \langle f_x^2 + f_y^2\rangle g^2\}_n \, dx. \tag{16}$$

To minimize (16), the Euler-Lagrange equations

$$\frac{d}{dx} \{\langle f^2\rangle g'\}_n - \{\langle ff_{xx} + 2f_x^2 + f_y^2\rangle g\}_n = 0 \tag{17}$$

must be satisfied. It is readily seen that

$$\langle f_n^2\rangle = y_b/h, \qquad \{\langle ff_{xx} + 2f_x^2 + f_y^2\rangle\}_n = c_n/hy_b\ , \tag{18}$$

where c_n is a constant which depends on the order n of the polynomial. The differential equation (17) thus assumes the form

$$[(1 - x/b)g_n']' - c_n g_n/h^2(1 - x/b) = 0 \tag{19}$$

and has the solution

$$g_n = (1 - x/b)^{\nu_n}, \qquad \nu_n = b(c_n)^{1/2}/h. \tag{20}$$

In particular for $n = 0$ we obtain

$$\langle f_0^2\rangle = 1 - \frac{x}{b}\,, \qquad \langle f_0 f_{0xx} + 2f_{0x}^2 + f_{0y}^2\rangle = \left(\frac{5}{2} + \frac{3}{2}\frac{h^2}{b^2}\right)\Big/ h^2\left(1 - \frac{x}{b}\right) \tag{21}$$

and

$$\nu_0^2 = \frac{5}{2}\left(\frac{b}{h}\right)^2 + \frac{3}{2}\,. \tag{22}$$

In first approximation ν_0 is the zeroth eigenvalue, $g_0 f_0$ the zeroth eigenfunction. For the special case $b/h = 10$ there results

$$\nu_0 = 15.9. \tag{23a}$$

This compares with the well-known precise solution

$$\nu_0 = \frac{\pi}{2 \operatorname{arccot} b/h} = 15.8 \tag{23b}$$

of the sector of vertex angle $2\alpha = 2 \operatorname{arccot} 10$.

3. The potential problem in a semi-infinite strip. The extremely simple problem $\nabla^2\Phi = 0$ in the semi-infinite strip of Fig. 2, with the boundary conditions

$$\begin{aligned} \Phi_{\mathrm{I}} &= \text{prescribed} \\ \Phi_\infty &= 0 \\ \Phi_{\mathrm{II}} &= \Phi_{\mathrm{III}} = 0 \end{aligned} \tag{24a,b,c}$$

provides a simple basis for comparing the polynomial method with the exact solution.

The same polynomials f_n are used here as in Section 2, one must only place

$$y_b = h = 1 \tag{25}$$

in Eqs. (12). The g_n functions are solutions

$$g_n = e^{-\nu_n x} \tag{26a}$$

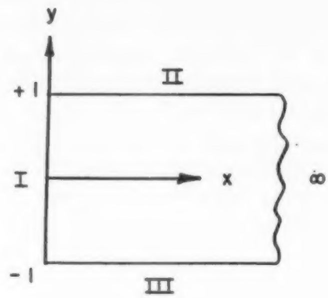

FIG. 2.

of the Euler equations

$$g_n'' - \nu_n^2 g_n = 0. \tag{26b}$$

The resulting eigenvalues are

$$\nu_n = \langle f_n'^2 \rangle^{1/2}; \qquad \nu_0 = 1.58, \nu_1 = 3.24, \nu_2 = 5.05, \cdots \tag{27}$$

as contrasted with the exact values

$$\frac{n+1}{2}\pi = 1.57, 3.14, 4.71, \cdots \tag{28}$$

For the specific example

$$\Phi_{\mathrm{I}} = 1 - y^4 \tag{29}$$

the exact solution becomes

$$\varphi_E = 1.173e^{-\pi x/2} \cos\frac{\pi y}{2} - 0.209e^{-3\pi x/2} \cos\frac{3\pi y}{2} + \cdots \tag{30}$$

while the polynomial method leads to

$$\varphi_A = 1.143e^{-1.58x}(1 - y^2) - 0.143e^{-5.05x}(1 - 8y^2 + 7y^4) \tag{31}$$

In contrast, a one-term Rayleigh approximation

$$\varphi_R = e^{-\nu x}(1 - y^4) \tag{32}$$

gives

$$\varphi_R = e^{-1.79x}(1 - y^4). \tag{33}$$

It is interesting to make a numerical comparison of the solutions E, A, R. At the point $(x, y) = (1, 0)$ they give, respectively

$$\varphi_E = 0.242, \qquad \varphi_A = 0.235, \qquad \varphi_R = 0.167. \tag{34}$$

Function (29) can be considered as the zeroth orthogonal polynomial constructed from the set $1, y^4, y^8, \cdots$. Clearly, the resulting approximation is poorer than when the set $1, y^2, y^4, \cdots$ is used. Since both sets are complete[7] there arises the question of

[7]By Szász' Theorem (Courant-Hilbert, *Methoden der Mathematischen Physik*, I. p. 86) the set $1, y_{\lambda_1}, y_{\lambda_2}, \cdots$ is complete, in a finite interval, 0 to h, whenever $\Sigma\ (\lambda_k)^{-1}$ diverges.

the proper selection of the best set. The answer is: the best set is obtained when the gap between successive powers is the smallest, consistent with boundary, symmetry and finiteness requirements for the functions and their derivatives.

4. The two-region problem. Consideration of a simple two-region potential problem (Fig. 3) shows an additional advantage in the use of the polynomial method, namely that the often troublesome transcendental boundary equations are replaced by a set of equations which are linear in the unknowns. Consider $\nabla^2\Phi = 0$ in each of two regions, A and B, with boundary conditions:

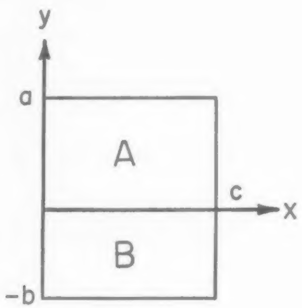

FIG. 3.

$$\Phi = 0 \quad \text{on the edges} \quad y = a,\ y = -b \quad \text{and} \quad x = c \tag{35a}$$

$$\Phi = \text{arbitrary on edge} \quad x = 0 \tag{35b}$$

$$\Phi_A = \Phi_B\ ,\ k\Phi_{Ay} = \Phi_{By}\ ,\ \Phi_{Ayy} = \Phi_{Byy}\ ,\ k\Phi_{Ayyy} = \Phi_{Byyy}\ ,\ \cdots \tag{35c}$$

on the interface, $y = 0$. The polynomials $f_n(y)$ to be used have a different form in each of the two regions A and B:

$$f_{A0} = (y - a)(A_{00} + A_{01}y),\ f_{B0} = (y + b)(B_{00} + B_{01}y),$$

$$f_{A1} = (y - a)(A_{10} + A_{11}y + A_{12}y^2),\ f_{B1} = (y + b)(B_{10} + B_{11}y + B_{12}y^2),\ \cdots \tag{36}$$

The four constants in f_0 are determined from the normalization condition and three interface conditions. The six constants in f_1 are determined from the two conditions of orthonormality, and four interface conditions. Successive polynomials are constructed similarly. The appropriate orthonormalization condition is

$$\int_{-b}^{0} f_{Bn}f_{Bm}\,dy + \int_{0}^{a} f_{An}f_{Am}\,dy = \delta_{nm}\ . \tag{37}$$

For the particular values of the parameters

$$k = 3(2)^{1/2}, \qquad a = 1, \qquad b = 1/2^{1/2} \tag{38a}$$

one finds

$$\nu_0 = \langle f_0'^2 \rangle^{1/2} = 1.75. \tag{38b}$$

This compares with the exact value

$$\nu_0 = 1.67 \tag{39a}$$

obtained by solving

$$k \tan \nu b + \tan \nu a = 0. \tag{39b}$$

5. The potential problem in a sector. The case of the circular sector brings out a novel feature in the orthogonality condition. The geometry is as in Fig. 4. Here the

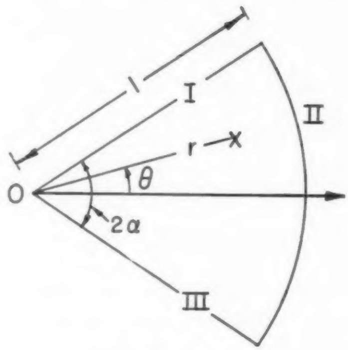

Fig. 4.

integral to be minimized for the k-th approximate eigenfunction

$$\varphi_k = g_k(r) f_k(\theta) \tag{40}$$

is

$$\begin{aligned} I_k &= \int_0^1 \int_{-\alpha}^{+\alpha} [(\partial \varphi_k/\partial r)^2 + (\partial \varphi_k/r\, d\theta)^2]\, d\theta\, r\, dr \\ &= \int_0^1 \int_{-\alpha}^{+\alpha} [(g_k' f_k)^2 + (g_k f_k'/r)^2]\, d\theta\, r\, dr. \end{aligned} \tag{41}$$

For cases where boundary values along the circular arc are specified, we are led to θ polynomials of type f_n considered in Section 3.

For the sake of variety we list the f_n polynomials appropriate to the conditions of vanishing normal edge derivative:

$$[\partial \Phi / r\, d\theta]_{\pm \alpha} = 0. \tag{42}$$

They are

$$\begin{aligned} f_0 &= 1/(2\alpha)^{1/2}, \\ f_1 &= (315/316\alpha)^{1/2}(\theta/\alpha)[1 - \tfrac{1}{3}\theta^2/\alpha^2], \\ f_2 &= (343/384\alpha)^{1/2}[1 - (30/7)(\theta/\alpha)^2 + (15/7)(\theta/\alpha)^4], \cdots \end{aligned} \tag{43}$$

When radial boundary values need representation, then an expansion in polynomials of r becomes necessary. On denoting

$$\langle v(r) \rangle \equiv \int_0^1 v(r)\, dr \tag{44a}$$

and integrating out over the variable r in (41),

$$I_k = \int_{-\alpha}^{+\alpha} [f_k'^2 \langle g_k^2/r \rangle + f_k^2 \langle r g_k'^2 \rangle] \, d\theta \tag{44b}$$

we are lead to the orthonormalization condition

$$\langle g_k g_l / r \rangle \equiv \int_0^1 (g_k g_l / r) \, dr = \delta_{kl} \, . \tag{45}$$

We list the functions $g_k(r)$ for the boundary conditions

$$\partial \Phi_{\mathrm{I}} / \partial r = \partial \Phi_{\mathrm{III}} / \partial r = R(r) = \text{prescribed function of } r, \tag{46a}$$

$$\Phi = 0 \qquad \text{at} \qquad r = 1, \tag{46b}$$

$$\Phi = \Phi / r = 0 \qquad \text{at} \qquad r = 0. \tag{46c,d}$$

Condition (46d), which essentially states that the polynomials $g_k(r)$ must contain a factor r^2, results from the requirement that in the corner, $r \to 0$, of the sector the transverse variation, $\partial \Phi / r \, \partial \theta$, of the function must be limited (otherwise Φ would not be uniquely determined).

The functions are

$$\begin{aligned} g_1 &= 60^{1/2} r^2 (1 - r), \\ g_2 &= 40^{1/2} r^2 (1 - r)(4 - 7r), \\ g_3 &= 140^{1/2} r^2 (1 - r)(5 - 20r + 18r^2), \\ g_4 &= 840^{1/2} r^2 (1 - r)(4 - 27r + 54r^2 - 33r^3), \cdots \end{aligned} \tag{47}$$

In either case, whether we are concerned with expansions into $g_n(r)$ functions along boundaries I, III, or expansions into $f_n(\theta)$ functions along boundary II, the remaining procedure is the same—insert the polynomials into the integral (41), carry out the integration over the proper variable, and solve the appropriate Euler-Lagrange equation for the second (unknown) factor of the eigenfunction.[8]

In Fig. 5 we plot successive approximations

$$R(r) \sim \sum a_n g_n(r), \qquad a_n \equiv \langle R g_n / r \rangle \tag{48a}$$

to the function

$$R = r \tag{48b}$$

in terms of the functions (47). Since we now have to construct a g_k expansion for a function $R(r)$ which does not satisfy the boundary conditions prescribed for $g_k(r)$, the approximation exhibits a pronounced Gibbs phenomenon near the end, $r = 1$, of the interval.

---

[8]Solution of the general problem, when non-homogeneous boundary conditions are specified along all edges, is obtained by superposition.

6. The wave equation in a rectangle. The second approximation. In this section we shall treat the vibration problem of a membrane

$$\nabla^2\Phi = K^2\Phi^{\cdot\cdot} \tag{49}$$

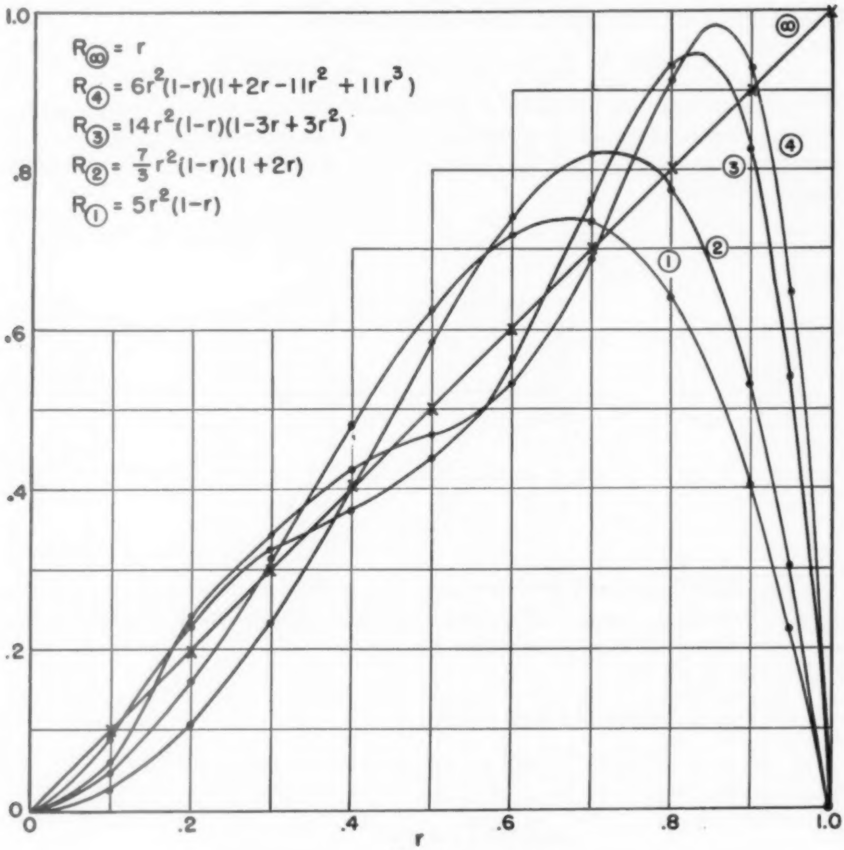

FIG. 5.

in a simple rectangle region. The example will illustrate the use of orthogonal polynomials for a problem where more than two independent variables are involved; it will also demonstrate the use of the "second approximation."

Let the problem be as follows: Solve (49) for $\Phi(x, y, t)$ in the rectangular region, Fig. 6, subject to the initial condition

$$\Phi(x, y, 0) = \text{prescribed} \tag{50}$$

and to the following boundary conditions (at all t):

$$\Phi_{\mathrm{II}} = \Phi_{\mathrm{IV}} = 0, \tag{51a}$$

$$\Phi_{\mathrm{I}} = \partial\Phi_{\mathrm{III}}/\partial x = 0. \tag{51b}$$

Problem (49) is equivalent to the requirement that one minimize the integral:

$$I = \int_0^\infty \int_0^b \int_{-a}^{a} [(\partial\Phi/\partial x)^2 + (\partial\Phi/\partial y)^2 - K^2(\partial\Phi/\partial t)^2]\, dy\, dx\, dt \tag{52}$$

subject to (50), (51). We write in "first approximation"

$$\Phi(x, y, t) \sim \sum C_{kl}\varphi_{kl}(x, y, t) = \sum C_{kl} g_k(x) f_l(y) \tau_{kl}(t), \tag{53}$$

where $g_k(x)$ and $f_l(y)$ are members of appropriate sets of orthogonal polynomials, the functions τ_{kl} will be determined by solving Euler's equations for the time variable after the integration over x and y has been carried out, and the C_{kl} are to be determined from the initial conditions.

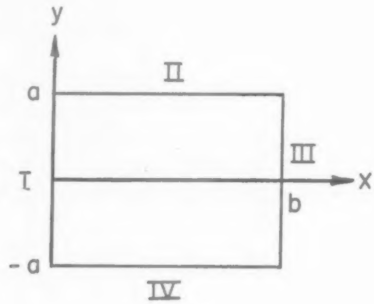

Fig. 6.

The $f_l(y)$ functions should go to zero at $y = \pm a$; thus they are the same polynomials as those of Section 2, except that y_b and h are now replaced by a. The $g_k(x)$ must be zero for $x = 0$ and have zero derivative for $x = b$.
This gives rise to the set

$$\begin{aligned} g_0 &= (15/2b)^{1/2}(x/b)[1 - x/b], \\ g_1 &= (273/2b)^{1/2}(x/b)[1 - (61/26)(x/b) + (16/13)(x/b)^2], \cdots \end{aligned} \tag{54}$$

Now,

$$\begin{aligned} \langle g_k g_l \rangle &\equiv \int_0^b g_k g_l \, dx = \delta_{kl} , \\ \langle f_k f_l \rangle &\equiv \int_{-a}^{+a} f_k f_l \, dy = \delta_{kl} , \end{aligned} \tag{55a,b}$$

and, in first approximation,

$$\begin{aligned} \langle g_k' g_l' \rangle &= \langle g_k'^2 \rangle \delta_{kl} , \\ \langle f_k' f_l' \rangle &= \langle f_k'^2 \rangle \delta_{kl} , \end{aligned} \tag{56a,b}$$

so that the integral (52) to be minimized becomes

$$I = \sum_{k,l} I_{kl} , \tag{57a}$$

$$I_{kl} = \int_0^\infty [\{\langle g_k'^2 \rangle + \langle f_l'^2 \rangle\} \tau_{kl}^2 - K^2 \dot{\tau}_{kl}^2]\, dt. \tag{57b}$$

The integrals (57b) lead to the Euler-Lagrange equations

$$\ddot{\tau}_{kl} + \nu_{kl}^2 \tau_{kl} = 0, \tag{58a}$$

where

$$K^2 \nu_{kl}^2 = \langle g_k'^2 \rangle + \langle f_l'^2 \rangle. \tag{58b}$$

Assuming that initial conditions require that

$$\tau_{kl}(0) = 1, \qquad \dot{\tau}_{kl}(0) = 0 \tag{59a}$$

it follows that

$$\tau_{kl} = \cos \nu_{kl} t \tag{59b}$$

The first two values of $\langle g_k'^2 \rangle^{1/2}$ and first six values of $\langle f_l'^2 \rangle^{1/2}$ are listed below and compared with the appropriate true eigenvalues $(k + \frac{1}{2})\pi$ and $(l + 1)\pi/2$ respectively.

k	$b\langle g_k'^2 \rangle^{1/2}$	$(k + 1/2)\pi$	% error
0	1.58	1.57	0.6
1	4.85	4.71	3.0
l	$a\langle f_l'^2 \rangle^{1/2}$	$(l + 1)\pi/2$	% error
0	1.58	1.57	0.6
1	3.24	3.14	3.2
2	5.05	4.71	7.2
3	7.04	6.28	12
4	9.19	7.85	16
5	11.51	9.42	22

One arrives at the second approximation ψ_{kl} by writing, in accordance with (5a, b),

$$\psi_{kl}(x, y, t) = g_{k-1}T_{k-1,l}f_l + g_kT_{k,l-2}f_{l-2} + g_kT_{kl}f_l + g_kT_{k,l+2}f_{l+2} + g_{k+1}T_{k+1,l}f_l\,, \tag{60a}$$

$$\psi_{kl}(x, y, 0) = g_k(x)f_l(y). \tag{60b}$$

These functions, orthonormal at $t = 0$, are suitable for expanding the initial value (50) of Φ. To simplify the notation we shall consider the special case of

$$\psi_{03} = g_0(T_{01}f_1 + T_{03}f_3 + T_{05}f_5) + g_1T_{13}f_3 \tag{61}$$

and to further simplify the calculations we shall neglect the last term, $g_1T_{13}f_3$ of this expression. Insertion of (61) into the variational integral (52) and minimization leads to the equations of motion

$$\begin{aligned}
[\langle g_0'^2 \rangle + \langle f_1'^2 \rangle]T_{01} + \langle f_1'f_3' \rangle T_{03} + \langle f_1'f_5' \rangle T_{05} + K^2\ddot{T}_{01} &= 0,\\
\langle f_1'f_3' \rangle T_{01} + [\langle g_0'^2 \rangle + \langle f_3'^2 \rangle]T_{03} + \langle f_3'f_5' \rangle T_{05} + K^2\ddot{T}_{03} &= 0,\\
\langle f_1'f_5' \rangle T_{01} + \langle f_3'f_5' \rangle T_{03} + [\langle g_0'^2 \rangle + \langle f_5'^2 \rangle]T_{05} + K^2\ddot{T}_{05} &= 0.
\end{aligned} \tag{62}$$

For a natural mode of vibration

$$T_{0i} = T_i \cos \nu t, \tag{63a}$$

$$K^2\nu^2 = \langle g_0'^2 \rangle + \alpha^2 = (5/2b)^2 + \alpha^2, \tag{63b}$$

the secular equation system becomes

$$\begin{aligned} \left(\frac{21}{2} - \alpha^2\right)T_1 - \frac{3}{2}\,11^{1/2}T_3 + \frac{5}{4}\,6^{1/2}T_5 &= 0 \\ -\frac{3}{2}\,11^{1/2}T_1 + \left(\frac{99}{2} - \alpha^2\right)T_3 - \frac{15}{4}\,66^{1/2}T_5 &= 0 \\ \frac{5}{4}\,6^{1/2}T_1 - \frac{15}{4}\,66^{1/2}T_3 + \left(\frac{265}{2} - \alpha^2\right)T_5 &= 0 \end{aligned} \tag{64}$$

This yields the frequency equation

$$-\alpha^6 + 192.5\alpha^4 + 7507.5\alpha^2 + 56306.25 = 0, \tag{65}$$

with solutions

$$\alpha_1^2 = 3.141595^2, \qquad \alpha_3^2 = 6.32440^2, \qquad \alpha_5^2 = 11.94288^2. \tag{66}$$

Assigning now to α^2 in (64) successively each of the values (66), and solving the corresponding systems, we obtain

$$\begin{aligned} {}^1T_1 : {}^1T_3 : {}^1T_5 &= 1 : 0.131442 : 0.007686, \\ {}^3T_1 : {}^3T_3 : {}^3T_5 &= -0.134007 : 1 : 0.333781, \\ {}^5T_1 : {}^5T_3 : {}^5T_5 &= 0.035561 : -0.329016 : 1, \end{aligned} \tag{67}$$

where the superscript of T refers to the particular frequency α_i^2 with which the amplitude triplet ${}^iT_1$, ${}^iT_3$, ${}^iT_5$ is associated. Consequently T_{0i} has the form

$$T_{0i} = A_1\,{}^1T_i \cos(\nu_1 t + \gamma_1) + A_3\,{}^3T_i \cos(\nu_3 t + \gamma_3) + A_5\,{}^5T_i \cos(\nu_5 t + \gamma_5) \tag{68}$$

The six constants A_1 , $\cdots$, γ_5 are to be determined from the six initial conditions

$$\begin{aligned} T_{01}(0) = 0, \qquad T_{03}(0) = 1, \qquad T_{05}(0) = 0, \\ \dot{T}_{01}(0) = \dot{T}_{03}(0) + \dot{T}_{05}(0) = 0. \end{aligned} \tag{69}$$

One obtains

$$A_1 = 0.12920, \quad A_3 = 0.88545, \quad A_5 = -0.29654, \quad \gamma_1 = \gamma_3 = \gamma_5 = 0. \tag{70}$$

This then completely determines the function ψ_{03} of (61) (if we neglect, as proposed, the T_{13} term). Separating out from the expression of ψ_{03} the coefficient of $\cos \nu_{03}t$ one is led to the second approximation, φ_{03}^* , of the true second eigenfunction Φ_{03} (in the approximation that the $g_1f_3T_{13}$ contribution is neglible):

$$\varphi_{30}^* = Ng_0({}^3T_1f_1 + {}^3T_3f_3 + {}^3T_5f_5) \cos \nu_{30}t$$

$$= N(x/b)(1 - x/b)(y/a)(1 - y^2/a^2)[1.941 + 1.654(y/a)^2 - 18.719(y/a)^4]$$

$$\cdot \cos \{[(5/2b)^2 + (6.32/a)^2]^{1/2}t/K\} \qquad (71)$$

(N is the normalization factor.)

Since the expansion of the initial value $\Phi(x, y, 0)$ of Φ in the functions (60b) is perfectly straightforward (and hence the present procedure does provide one avenue of improved solution), one may ignore, until further study has been made, the question as to what is preferable: Use simple second approximation functions of type (61) which —at $t = 0$—violate slightly the orthogonality condition

$$\iint \Phi_{kl}(x, y)\Phi_{k'l'}(x, y)\, dy\, dx = \delta_{kk'}\delta_{ll'} \qquad (72)$$

or use the conventional approach involving an infinite secular system,[9] and extract from this system a set of unwieldy but precisely orthogonal approximate eigenfunctions. These precisely orthogonal functions will be successive approximations to the true eigenfunctions according to whether we retain in the infinite secular system only terms along the principal diagonal, also terms along the adjacent sub- and superdiagonals, also terms along the next sub- and superdiagonals, etc.

[9]See e.g. *Eigenwertaufgaben*, by L. Collatz, Akad. Verlagsges., 1949, p. 398.

—NOTES—

A SUGGESTED MODIFICATION OF NOISE THEORY*

By JULIAN KEILSON (*Massachusetts Institute of Technology*)

Abstract. A class of stationary, equilibrium, Markoff processes is demonstrated all of which have the same equilibrium distribution, $W_0(x)$, and correlation function $R(t) = E_0 \exp(-t/\tau_0)$ differing from each other in the number of zero crossings of the system per second. The processes are described by an integral equation characterized by a parameter γ. As γ approaches 1, the integral equation passes over into the Fokker-Planck equation

$$\tau_0 \frac{\partial W}{\partial t} = E_0 \frac{\partial^2 W}{\partial x^2} + \frac{\partial}{\partial x}(xW).$$

Since the number of zero crossings per second of the system becomes infinite as γ goes to one, the degenerate nature of the Fokker-Planck process is made evident.

1. Introduction. Any stationary Markoffian motion of a system in one dimension may be described by an equation of the form

$$\frac{\partial W(x, t)}{\partial t} = -W(x, t) \int A(x, x')\, dx' + \int W(x', t) A(x', x)\, dx', \tag{1}$$

where $W(x, t)dx$ is the probability of finding the system in the interval $(x, x + dx)$ at time t. $A(x, x')dx'$ is the probability per unit time that the system if at x, will jump to the interval $(x', x' + dx')$. Equation (1) describes the manner in which changes in local probability density can occur. It is seen that

$$\frac{d}{dt} \int W(x, t)\, dx = 0 \tag{2}$$

so that probability is conserved.

The stationary Markoff character of the motion is assured by the formulation in terms of a time independent transition function $A(x, x')$. By stationary Markoff is meant that if at $t = 0$, the system is at x_0, its subsequent distribution is completely described by the second order conditional probability:

$$W(x, t) = P_2(x_0 \mid x; t). \tag{3}$$

For some $A(x, x')$ the process will be an equilibrium process in the sense that

$$\lim_{t \to \infty} P_2(x_0 \mid x; t) = W_0(x) \tag{4}$$

is independent of x_0.

If $A(x, x')$ is not localized about $x = x'$ the stochastic motion described is discontinuous. The system jumps about from point to point. As $A(x, x')$ becomes more localized about $x = x'$, the jumps become on the average smaller, i.e., the motion becomes "more continuous".

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In this paper a class of transition function $A_\gamma(x, x')$ will be studied. It will be shown that, as $\gamma \to 1$ and $A_\gamma(x, x')$ becomes increasingly localized about $x = x'$, the integral equation (1) passes over into the Fokker-Planck equation

$$\tau_0 \frac{\partial W}{\partial t} = E_0 \frac{\partial^2 W}{\partial x^2} + \frac{\partial}{\partial x}(xW). \tag{5}$$

The nature of the motion described by the Fokker-Planck equation as indicated by this limiting process will be discussed, and the physical implications of this study examined.

2. Properties of the integral equation. Consider the process described by the transition function

$$A(x, x') = \frac{1}{T_{mf}} (\beta/\pi)^{1/2} \exp\left[-\beta(x' - \gamma x)^2\right]. \tag{6}$$

Here, T_{mf}, the mean free time of the system, is independent of x, i.e.

$$\int_{-\infty}^{+\infty} A(x, x')\, dx' = 1/T_{mf} \tag{7}$$

is the mean number of transitions of the system per unit time; β describes the dispersion occurring at each transition. The larger β, the more "continuous" is the motion; γ is a parameter describing the relaxation or damping of the system, i.e. if $\langle x \rangle$ is the mean value of x,

$$\langle x \rangle = \int xW(x, t)\, dx \tag{8}$$

then

$$\frac{d\langle x \rangle}{dt} = \frac{-\langle x \rangle}{T_{mf}/(1 - \gamma)} \tag{9}$$

as is readily deduced from Eq. (1). The decay time of system is, then

$$\tau = T_{mf}/(1 - \gamma) \tag{10}$$

The closer γ is to one, the smaller is the damping and the longer the decay time. τ is also the correlation time of the system, for

$$R(t) = \int W_0(x_0)x_0 P_2(x_0 \mid x; t)x\, dx_0\, dx.$$

Since from (9) $\langle x \rangle = x_0 \exp(-t/\tau)$, we have

$$R(t) = \int W_0(x_0)x_0^2 \exp(-t/\tau)\, dx_0\,,$$

i.e.

$$R(t) = E \exp(-t/\tau), \tag{11}$$

where

$$E = \int W_0(x)x^2\, dx. \tag{12}$$

Equation (1) with $A(x, x')$ given by (6) may be solved in the following way. Use is made of the expansion

$$\exp\left[-(x - \gamma y)^2/(1 - \gamma^2)\right] = \{[\pi(1 - \gamma^2)]^{1/2} \exp[+(y^2 - x^2)/2]\}.$$

$$\sum_0^\infty \gamma^n \psi_n(x)\psi_n(y), \tag{13}$$

where $\psi_n(x)$ is the set of orthonormal Hermite functions

$$\psi_n(x) = \left(\frac{1}{2^n n! \pi^{1/2}}\right)^{1/2} \exp(x^2/2)(-d/dx)^n \exp(-x^2). \tag{14}$$

Then

$$A(x', x) = \frac{\alpha}{T_{mf}} \exp\left[-\frac{\alpha^2}{2}(x^2 - x'^2)\right] \sum_0^\infty \gamma^n \psi_n(\alpha x)\psi_n(\alpha x'), \tag{15}$$

where

$$\alpha = [\beta(1 - \gamma^2)]^{1/2}.$$

In terms of

$$\phi(x, t) = \exp\left[\frac{\alpha^2}{2} x^2\right] W(x, t)$$

Eq. (1) becomes

$$\frac{\partial\phi}{\partial t} = -\frac{\phi}{T_{mf}} + \frac{\alpha}{T_{mf}} \int \sum_0^\infty \psi_n(\alpha x)\psi_n(\alpha x')\gamma^n \phi(x', t)\, dx'.$$

If we expand ϕ in terms of the orthonormal set $\alpha^{1/2}\psi_n(\alpha x)$

$$\phi(x, t) = \sum_0^\infty a_n \alpha^{1/2} \psi_n(\alpha x),$$

our equation separates into

$$\frac{da_n}{dt} = -\frac{a_n}{T_{mf}} + \frac{\gamma^n a_n}{T_{mf}}. \tag{16}$$

If, at $t = 0$,

$$W(x, 0) = \delta(x - x_0) = \sum_0^\infty \alpha \psi_n(\alpha x)\psi_n(\alpha x_0),$$

then

$$a_n(0) = \alpha^{1/2} \exp\left(\frac{\alpha^2}{2} x_0^2\right)\psi_n(\alpha x_0),$$

and

$$P_2(x_0 \mid x; t) = \exp\left[\frac{\alpha^2}{2}(x_0^2 - x^2)\right] \sum_0^\infty \{\alpha\psi_n(\alpha x)\psi_n(\alpha x_0) \exp[-t(1 - \gamma^n)/T_{mf}]\}. \tag{17}$$

We see that

$$\lim_{t\to\infty} P_2(x_0 \mid x; t) = W_0(x) = \pi^{-1/2}\alpha \exp(-\alpha^2 x^2)$$
$$= \pi^{-1/2}[\beta(1-\gamma^2)]^{1/2} \exp[-\beta(1-\gamma^2)x^2] \tag{18}$$

and

$$E = \langle x^2 \rangle = [2\beta(1-\gamma^2)]^{-1}. \tag{19}$$

3. Passage to the Fokker-Planck process. The Fokker-Planck equation, (5), contains two parameters, the correlation time τ_0 and the equilibrium mean square, $E_0 = \langle x^2 \rangle$.

Our transition function $A(x, x')$ contains three variables τ, E, γ. Let us maintain the values $\tau = \tau_0$, and $E = E_0$ and permit γ to pass through a set of values approaching one. To do so we adjust T_{mf} and β to the value of γ through the equations

$$T_{mf} = \tau_0(1-\gamma) \tag{20}$$

and

$$\beta = \frac{1}{2E_0(1-\gamma^2)} \simeq \frac{1}{4E_0(1-\gamma)}. \tag{21}$$

These parameter values define a set of processes A_γ all of which will have the same equilibrium distribution and the same correlation function as the corresponding Fokker-Planck process.

Indeed, the corresponding Fokker-Planck process is just the limit of the process A_γ as $\gamma \to 1$.

This may be seen in two ways. First one may pass to the limit $\gamma = 1$ in the conditional probability function.

Since for the process A_γ

$$\alpha = \left(\frac{E_0}{2}\right)^{1/2}, \qquad \lim \frac{1-\gamma^n}{T_{mf}} = \frac{n}{\tau_0},$$

we have

$$\lim_{\gamma\to 1} P_{2\gamma}(x_0 \mid x; t) = \exp\left[\frac{\alpha^2}{2}(x_0^2 - x^2)\right] \sum_0^\infty \{\alpha\psi_n(\alpha x)\psi_n(\alpha x_0) \exp(-nt/\tau_0)$$
$$= \{2E_0\pi[1-\exp(-2t/\tau_0)]\}^{-1/2} \exp\left\{\frac{-[x - x_0\exp(-t/\tau_0)]^2}{2E_0[1-\exp(-2t/\tau_0)]}\right\} \tag{22}$$

and this is indeed the second order conditional probability of our equation (5).

It can also be seen directly that the integral equation passes over into the Fokker-Planck equation.

The integral equation can always be rewritten formally* as a partial differential equation of infinite order

$$\frac{\partial W(x, t)}{\partial t} = \sum_1^\infty \frac{\partial^n}{\partial x^n}(A_n(x)W(x, t)), \tag{23}$$

*See p. 246 Keilson and Storer, Q. Appl. Math. 10, 243-253 (1952).

where

$$A_n(x) = \frac{1}{n!}\int (x - x')^n A(x, x')\, dx'. \tag{24}$$

In the limit $\gamma \to 1$, $A_1(x)$ and $A_2(x)$ do not vanish, but all higher moments do vanish. It is readily found that

$$A_1(x) = x/\tau_0$$

and

$$A_2(x) = \frac{(1-\gamma)^2x^2}{2T_{mf}} + \frac{(1-\gamma^2)E_0}{2T_{mf}} \to \frac{E_0}{\tau_0}.$$

The higher moments

$$A_n(x) = \frac{1}{n!}\int \{(1-\gamma)x + (\gamma x - x')\}^n \frac{1}{T_{mf}}\left(\frac{\beta}{\pi}\right)^{1/2} \exp\left[-\beta(x' - \bar{\gamma}x)^2\right] dx'$$

contain terms

$$(1-\gamma)^m x^m \langle(\gamma x - x')^p\rangle,$$

where either $m > 2$ or $p > 2$. A simple examination reveals that all such terms go to zero as γ approaches one.

4. **Zero crossings of the system A_γ.** It is easily seen that the number of times per second that the system with its motion characterized by $A(x, x')$ will cross zero is given by

$$\begin{aligned} j_+(0) &= \int_{-\infty}^0 W_0(x)\, dx \int_0^\infty A(x, x')\, dx' \\ &= \frac{(1-\gamma^2)^{1/2}}{\pi T_{mf}} \int_{-\infty}^0 \int_0^\infty \exp\left[-x^2 - y^2 - 2\gamma xy\right] dx\, dy \\ &= \frac{(1-\gamma^2)^{1/2}}{\pi T_{mf}} \int_A \exp\left[-S^2(1+\gamma) - T^2(1-\gamma)\right] ds\, dt, \end{aligned}$$

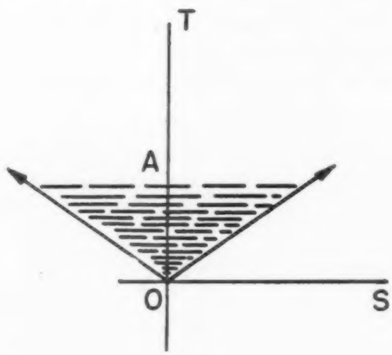

FIG. 1

where A is the shaded area shown in Fig. 1, i.e.

$$j_+(0) = \frac{1}{\pi T_{mf}} \tan^{-1} \left\{ \left(\frac{1 - \gamma}{1 + \gamma} \right)^{1/2} \right\}. \tag{25}$$

For the process A_γ ,

$$j_+(0) = \frac{1}{\pi \tau_0 (1 - \gamma)} \tan^{-1} \left\{ \left(\frac{1 - \gamma}{1 + \gamma} \right)^{1/2} \right\}. \tag{26}$$

As γ approaches one, $j_+(0)$ becomes infinite. This will be true not only for zero but for all x. The implication is plain. The Fokker-Planck process is a degenerate process in which the one sided current density of the system is infinite. A Fokker-Planck model for the velocity motion of a colloid particle would describe an infinite number of changes of direction of the particle per unit time. Such a model used to describe voltage fluctuations would imply an infinite number of polarity reversals per second. Since a process $A\gamma$ will afford the same correlation function and equilibrium distribution, *and* finite polarity reversal frequency, it is suggested that such a model may better describe noise, and that the number of zero crossings be regarded as an independent macroscopic physical quantity on an equal footing with τ_0 , E_0 .

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EVALUATION OF CONSTANTS IN CONFORMAL REPRESENTATION*

By SAMUEL I. PLOTNICK AND THOMAS C. BENTON (*Pennsylvania State University*)

In using the Schwarz-Christoffel transformation [1],

$$dz = K \prod_{i=1}^{n} (\zeta - \zeta_i)^{(\alpha_i/\pi)-1} d\zeta = Kf(\zeta)\, d\zeta$$

whereby the upper half ζ-plane is mapped into a simple connected polygon, the evaluation of the unknown constant K (if complex $K = ce^{i\lambda}$, c, λ real), is oftentimes tedious. We shall show a simple method of evaluating the unknown constant K by examples, proving first a

THEOREM: *By the Schwarz-Christoffel transformation if ζ_i in the ζ-plane corresponds to two points P_i , Q_i in the z-plane and $\zeta = \zeta_i$ is a simple pole of $f(\zeta)$, then*

$$K = \frac{\text{dist}\,(P_i\,,\,Q_i)}{\pi i R(\zeta = \zeta_i)}$$

R, denoting residue and dist $(P_i\,, Q_i)$, denoting the distance between the two points P_i and Q_i .

*Received May 8, 1953.

PROOF: Since

$$dz = Kf(\zeta)\,d\zeta$$

$$\int_{P_i}^{Q_i} dz = \text{dist}\,(P_i\,,\,Q_i)$$

$$= \lim_{\delta\to 0}\int_0^{\pi} Kf(\zeta_i + \delta e^{i\theta})i\delta e^{i\theta}\,d\theta$$

$$= K\lim_{\delta\to 0}\int_0^{\pi} f(\zeta_i + \delta e^{i\theta})i\delta e^{i\theta}\,d\theta,$$

where $\zeta - \zeta_i = \delta e^{i\theta}$.
Since f has a simple pole at ζ_i the Laurent expansion [2], is

$$f(\zeta) = \frac{R}{\zeta - \zeta_i} + g(\zeta)$$

with $g(\zeta)$ analytic near $\zeta = \zeta_i$. Now,

$$f(\zeta_i + \delta e^{i\theta}) = \frac{R}{\delta e^{i\theta}} + g(\zeta_i + \delta e^{i\theta}).$$

Therefore,

$$\lim_{\delta\to 0}\int_0^{\pi} f(\zeta_i + \delta e^{i\theta})i\delta e^{i\theta}\,d\theta = \lim_{\delta\to 0}\left[Ri\int_0^{\pi} d\theta + i\delta\int_0^{\pi}\sum_{k=0}^{\infty} c_k\delta^k e^{(k+1)i\theta}\,d\theta\right]$$

$$= i\pi R$$

whence

$$K = \frac{\text{dist}\,(P_i\,,\,Q_i)}{\pi i R(\zeta = \zeta_i)}.$$

Consider the transformation described by Milne-Thomson [3], whereby we map the infinite strip on the upper half ζ-plane. The Schwarz-Christoffel transformation gives

$$\frac{dz}{d\zeta} = K\zeta^{-1} \qquad \text{or} \qquad z = K\int\frac{d\zeta}{\zeta} + L.$$

$L = 0$ for $z = 0$ corresponds to $\zeta = 1$.
By the theorem,

$$K = \frac{\text{dist}\,(P_i\,,\,Q_i)}{\pi i R(\zeta = 1)} = \frac{ai}{\pi i(1)} = \frac{a}{\pi}$$

In this case comparison of real and imaginary parts with infinities is avoided.

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A UNIQUE LAW FOR IDEAL INCOMPRESSIBLE FLOW WITH PRESERVED PATTERN OF FINITE SEPARATION*

By H. S. TAN (*Cornell University*)

Prandtl, in treating the rolled up separation surface for a flow around corner (Fig. 1 (a); Ref. 1), and von Kármán, in treating the wake formation for a flow normal to a

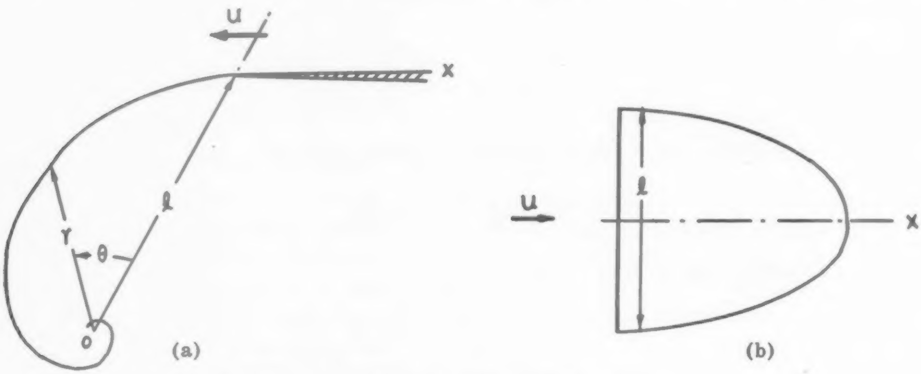

FIG. 1. (a) Prandtl's case of flow around a corner.
(b) von Kármán's case of flow past a flat plate.

flat plate (Fig. 1(b); Ref. 2), both arrived at a law of fluid motion with stationary flow pattern which can be reduced to the form

$$U(t) = \frac{U_0}{1 - (a_0 t/U_0)},$$

where we denote by $U(t)$ the velocity at $r = l$ for the first case, and at infinity for the second case, by a the acceleration at the same point; t is the time, and the subscript 0 indicates the initial condition.

It can readily be shown that this law of motion is in fact at once general and unique. The law is general in the sense that it applies to any body shape with well defined separation points, provided there exists a solution. The law is unique because no other fluid motion except this one will produce such a preserved flow pattern.

To prove these statements, let us observe that a satisfactory non-stationary velocity potential Φ must be of the form

$$\Phi(x, y, t) = U(t)\phi(x, y) \tag{1}$$

in order that the flow pattern may not alter with time. This leads directly to the following expression for the pressure:

$$p = \phi \frac{dU}{dt} + \frac{1}{2} U^2 \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right]. \tag{2}$$

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Now, ϕ is determined by the differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{3}$$

together with the boundary conditions

	plate flow	corner flow
i) at infinity	$\phi_x = 1$	$\phi_x = 0$
ii) on solid boundary	$\phi_n = 0$	$\phi_n = 0, \phi_x(l) = 1$
iii) on free boundary	$p = 0$	$\Delta_n p = 0$

The only boundary condition that involves U, and hence t, is (*iii*). Since determination of ϕ cannot depend on t, (*iii*) must be reducible to a product form, i.e.

$$L(U)M(\phi) = 0. \tag{4}$$

But this is evidently the case when and only when U satisfies the following differential equation

$$\frac{dU}{dt} = \pm kU^2, \qquad k > 0 \tag{5}$$

by virtue of (2).

Integration of (5) leads immediately to the desired law of fluid motion:

$$U(t) = \frac{U_0}{1 \mp kU_0t}, \qquad U_0 > 0, \tag{6}$$

where U_0 is the initial velocity at $r = l$ for Prandtl flow, and at infinity for Kármán flow. Introducing the initial acceleration at corresponding points a_0 , we see that (6) further reduces to

$$U(t) = \frac{U_0}{1 - (a_0t/U_0)} \tag{7}$$

which result has been plotted in Fig. 2.

It may be observed that a_0 is uniquely determined in terms of U_0 and a reference length l (where $\phi(l) = 1$ in Prandtl flow, and is the plate width in Kármán flow) by certain kinematic considerations. Indeed, for the case of Kármán's plate flow, the condition that the wake must form a closed region leads to the following value for initial acceleration at infinity

$$a_0 = 0.298\,\frac{U_0^2}{l}. \tag{8}$$

Whereas for the case of Prandtl's corner flow the condition that the rolled up vortex sheet must coincide with the stream surface leads to the following result

$$a_0 = \frac{\exp(2\pi 3^{1/2}) - 1}{2[1 + \exp(-\pi 3^{1/2})]}\,\frac{U_0^2}{l}. \tag{9}$$

From Fig. 2 it may be noticed that although it is entirely possible to produce a preserved pattern of finite separation both by an accelerated and a decelerated flow

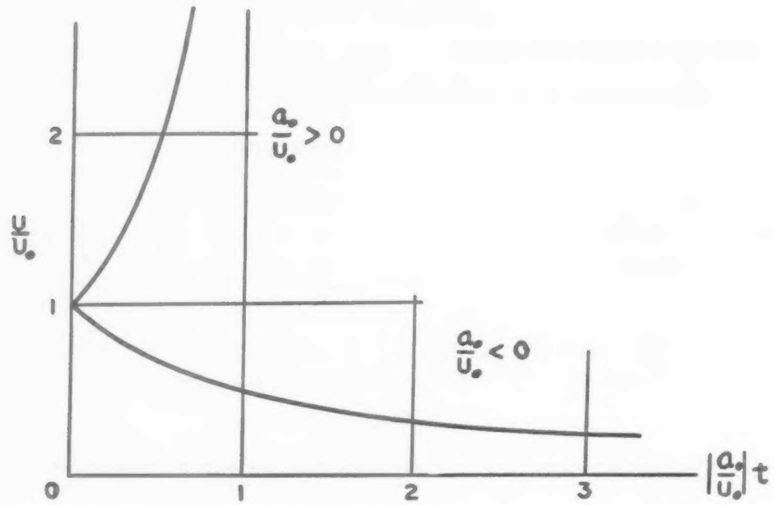

FIG. 2. Variation of flow speed U with time t, according to Eq. (7), for accelerated (above) and decelerated (below) flow.

field, yet to maintain such a flow pattern indefinitely, an accelerated field is never adequate; the flow then must be a retarded one.

This analysis definitely rules out the possibility of preserving a finite wake in a stationary flow field, or in any flow field that does not exactly follow the law of motion (7).

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THE BOUNDARY LAYER ON A QUARTER INFINITE FLAT PLATE*

BY L. TRILLING (*Massachusetts Institute of Technology*)

This note discusses the incompressible boundary layer on the surface of a quarter infinite flat plate in the absence of a pressure gradient, generalizing the classical two dimensional Blasius solution [1] and Sears' extension to an arbitrarily yawed plate [2]. It shows that the flow retains free stream direction and Blasius profile at all points of the plate, and that the projection of the constant velocity surfaces on planes parallel

*Received May 21, 1953.

to the plate are curves which become asymptotically parallel to the plate edges far from the lead corner so that the Blasius and Sears solutions are asymptotic cases of the solution given below.

Let the plate occupy the first quadrant of the X, Y plane with the lead corner at the origin; let the free stream velocity vector U_∞ be directed along the line $X - Y = 0$ (the results obtained in this case can easily be generalized to an arbitrary angle of approach by suitable stretching of the Y coordinate).

The equations satisfied by the flow in the resulting boundary layer are

$$U_X + V_Y + W_Z = 0, \tag{1a}$$

$$UU_X + VU_Y + WU_Z = \nu U_{ZZ}, \tag{1b}$$

$$UV_X + VV_Y + WV_Z = \nu V_{ZZ}, \tag{1c}$$

with the boundary conditions

$$U = W = V = 0 \qquad \text{on the plate,} \tag{2a}$$

$$U = V = 2^{-1/2}U_\infty \qquad \text{as} \qquad Z \to \infty. \tag{2b}$$

If one seeks solutions of the form suggested by Sears and satisfying the boundary condition (2b), namely with

$$U(X, Y, Z) = V(X, Y, Z), \tag{3}$$

Eqs. (1) become

$$U_X + U_Y + W_Z = 0, \tag{4a}$$

$$U(U_X + U_Y) + WU_Z = \nu U_{ZZ}. \tag{4b}$$

It is convenient to introduce the Blasius parabolic coordinates

$$x = Z(X\nu/U_\infty)^{-1/2}, \qquad y = Z(Y\nu/U_\infty)^{-1/2}, \tag{5a}$$

and the dependent variables

$$u(x, y) = \frac{2^{1/2}U}{U_\infty}, \qquad w(x, y) = \frac{2^{1/2}WZ}{\nu}. \tag{5b}$$

The equations of motion then become, with $r^2 = x^2 + y^2$,

$$-rw_r + w + \frac{1}{2}(x^3u_x + y^3u_y) = 0, \tag{6a}$$

$$wru_r - \frac{u}{2}(x^3u_x + y^3u_y) = r^2u_{rr}, \tag{6b}$$

with the boundary conditions

$$u(0) = \frac{w(0)}{r} = 0, \tag{7a}$$

$$\lim_{x,y\to\infty} u(x, y) = 1. \tag{7b}$$

Note that with $y = 0$, $\partial/\partial y = 0$, $r = x$, the flow becomes a Blasius flow since one has

$$xw_x - w + \frac{1}{2} x^3 u_x = 0, \tag{8a}$$

$$wxu_x - \frac{u}{2} x^3 u_x = x^2 u_{xx} . \tag{8b}$$

Differentiating (8b) and substituting for w, w_x from (8a, b), one obtains

$$(u''/u')' + u/2 = 0 \tag{8c}$$

which is equivalent to the Blasius equation.

To reduce system (6a, b) to system (8a, b) in one variable, one must find a parameter $s(x, y)$ which satisfies the conditions

$$s \frac{\partial}{\partial s} = r \frac{\partial}{\partial r} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} , \tag{9a}$$

$$s^3 \frac{\partial}{\partial s} = x^3 \frac{\partial}{\partial x} + y^3 \frac{\partial}{\partial y} . \tag{9b}$$

In polar coordinates, the condition (9a) is satisfied if

$$s = r\lambda(\theta), \tag{10a}$$

while (9b) is satisfied if $s = r\lambda(\theta)$ is a solution of

$$x^3 s_x + y^3 s_y = s^3 \tag{10b}$$

or, using (10a),

$$\lambda(\cos^4 \theta + \sin^4 \theta) - \lambda' \sin \theta \cos \theta = \lambda^3. \tag{10c}$$

In terms of the new variables

$$\alpha = \cos 4\theta, \qquad \mu = \lambda^2, \tag{11a}$$

Eq. (10c) becomes

$$\mu(3 + \alpha) + 2(1 - \alpha^2)\mu' = 4\mu^2. \tag{11b}$$

A particular solution of this equation is

$$\mu_0 = (1 - \alpha)/4. \tag{12}$$

The general solution can be obtained by substituting

$$\mu = F(\alpha)\mu_0(\alpha) \tag{13a}$$

into (11b). The function $F(\alpha)$ then satisfies the separable equation

$$2(1 + \alpha)F' = F(F - 1), \tag{13b}$$

so that one has:

$$F = [1 - c^2(1 + \alpha)^{1/2}]^{-1}, \tag{13c}$$

where C is an arbitrary constant. When $C = 0$, one finds the solution $\mu_0(\alpha)$. Substituting (13c) into (13a) and (11a) and then (10a)

$$s = (1/2)r \sin 2\theta[1 - \alpha \cos 2\theta]^{-1/2}. \tag{14}$$

Since $s(r, \theta)$ satisfies equations (10a, b), $u(s)$, $w(s)$ satisfy the Blasius Equation (8a, b). Writing s in terms of physical coordinates, we have

$$r = Z[U_\infty(x + y)/\nu xy]^{1/2}, \qquad \sin \theta = (x/x + y)^{1/2}, \qquad \cos \theta = (y/x + y)^{1/2}, \tag{15a}$$

$$s = 2Z(U_\infty/\nu)^{1/2}[(x + y) - a(x - y)]^{-1/2}. \tag{15b}$$

The constant velocity surfaces are the surfaces s = const. since $u = Bl(s)$. It remains to determine the constant $| a |$ in such a way that as $X/Y \to \infty$, s becomes proportional to $Z(u_\infty/\nu y)^{1/2}$ so that one obtains the Sears yawed plate solution. This is the case if $a = -1$, $(x > y)$; $a = 1$ $(x < y)$.

It follows that the present solution extends Sears' result up to the immediate corner of the plate with no change. The constant shear and boundary layer thickness lines are parallel to the axes up to the axis of symmetry. This indicates the existence of a narrow region there where the cross flow terms in the velocity Laplacian are not negligible and the boundary layer equations do not hold.

References

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2. W. R. Sears, *The boundary layer of yawed cylinders*, Journ. Aero. Sci. **15**, 49 (1948).

NOTE ON MAXIMUM SHOCK DEFLECTION*

By GARRETT BIRKHOFF AND JOHN W. WALSH (*Harvard University*)

The angle $\lambda = \lambda(M)$ of maximum shock deflection, for a given Mach number M of flow, is of interest in various applications. It gives the critical angle for *attached shocks* past a wedge [2, p. 53], and that for *jetless wedge collapse* [3]. We give here a new simple means of determining $\lambda(M)$, for ideal gases, which seem simpler than the usual one [1, §122].

We follow the notation of [2]. By formulas (4.3) and (3.3) of this reference,

$$q_{2n} = \frac{a^{*2}}{q_{1n}} \qquad \text{and} \qquad \frac{a^{*2}}{q_{1n}^2} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1}\frac{a^2}{q_{1n}^2} \tag{1}$$

in a polytropic gas, with $p + p_0 = A\rho^\gamma$.

Now suppose we are given the velocity q_1, relative to J, of one impinging stream, and the shock angle β between the impinging stream and the shock front, as in Fig. 1. Of course, β is not known a priori; we shall seek, by variation of β, that value of β which maximizes the deflection angle δ, and thus obtain the desired maximum deflection $\delta_{max} = \lambda$.

We suppose also that we know p, ρ in the impinging stream. Then the normal shock,

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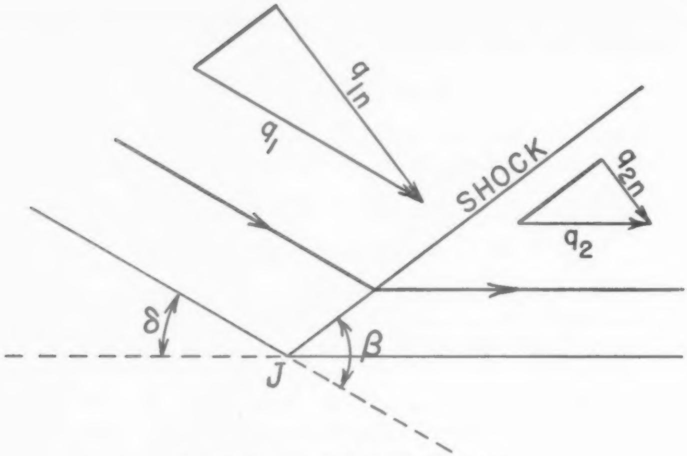

FIG. 1. Stream deflection by a shock.

associated as usual with the oblique shock by considering those velocity components that are normal to the shock front, has on the supersonic (approaching) side a velocity of

$$q_{1n} = q_1 \sin \beta. \tag{2}$$

Substituting from (2) and (1), we obtain the important formula

$$\frac{q_{2n}}{q_1} = \left[\frac{a^*}{q_{1n}}\right]^2 \frac{q_{1n}}{q_1} = \sin \beta \left[\frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1} \frac{a^2}{q_1^2} \csc^2 \beta\right]. \tag{3}$$

Equivalently, since $M_1^2 = q_1^2/a^2$,

$$\frac{q_{2n}}{q_1} = A \sin \beta + K \csc \beta, \tag{4}$$

where A and K are given by

$$A = (\gamma - 1)/(\gamma + 1), \qquad K = 2/M_1^2 (\gamma + 1), \tag{5}$$

for an ideal gas.

Using trigonometry, we now find that the component of q_2 *parallel* to q_1 is (cf. Fig. 1)

$$q_1 \cos^2 \beta + q_{2n} \sin \beta. \tag{6}$$

The component of q_2 *perpendicular* to q_1 is similarly

$$q_1 \cos \beta \sin \beta - q_{2n} \cos \beta. \tag{6'}$$

Taking the ratio, after dividing through the numerator and denominator by q_1, and using (4), we get

$$\tan \delta = \frac{\cos \beta \sin \beta - \cot \beta \,(A \sin^2 \beta + K)}{\cos^2 \beta + A \sin^2 \beta + K}\,. \tag{7}$$

We next rationalize the right side of (7), by the substitution $\tau = \cot \beta$. This gives $\csc^2 \beta = \tau^2 + 1$. Substituting in (6′) after dividing through the numerator and denominator by $\sin^2 \beta$, we get

$$\tan \delta = \tau\left[\frac{1 - A - K(\tau^2 + 1)}{\tau^2 + A + K(\tau^2 + 1)}\right] = \tau\left[\frac{E}{F + G\tau^2} - B\right], \tag{8}$$

where

$$B = K/(K + 1), \qquad E = (1 - A)/(K + 1), \qquad F = A + K,$$
$$G = K + 1. \tag{8'}$$

This is equivalent to [2, formula (4.26)]. This is, incidentally, a very convenient formula to use in computing shock polars.

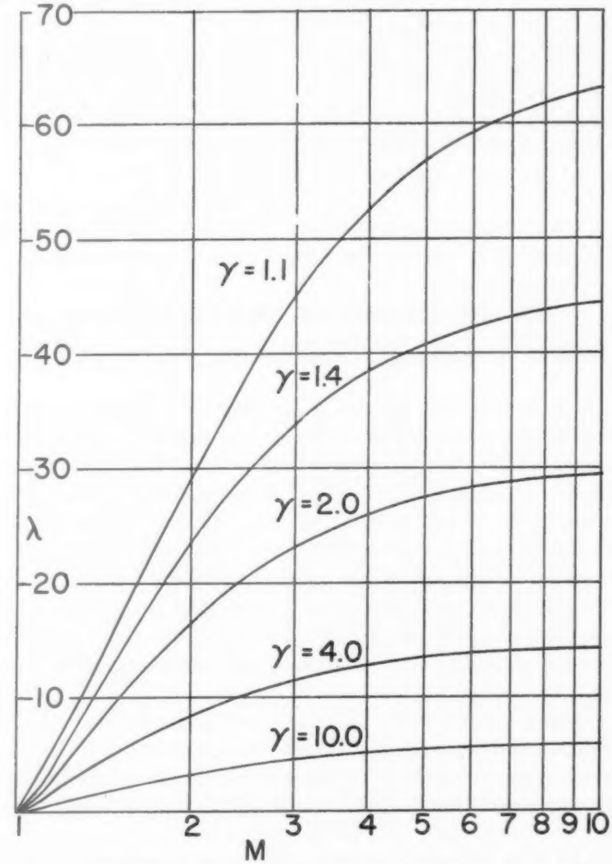

FIG. 2. Maximum stream deflection λ, as a function of γ and the Mach number M of the entry flow.

It is especially convenient in computing the *limiting angle* $\delta_{\max} = \lambda$. Since $\tan \delta$ is an increasing function, this is equivalent to maximizing the right-hand side of (8). But this maximum may easily be found by setting the derivative equal to zero. The derivative is

$$-B + \frac{E}{F + G\tau^2} - \frac{2EG\tau^2}{(F + G\tau^2)^2}\,.$$

Multiplying through by the denominator, we see that this derivative vanishes if and only if τ^2 satisfies

$$BG^2\tau^4 + (2BFG + EG)\tau^2 + (BF^2 - EF) = 0. \tag{9}$$

Our interest is confined to $M > 1, \gamma > 1$. Then it may easily be shown, using (8′), that the two roots of the equation are real; further, one of these roots is negative and has no physical significance since it leads to an imaginary value of τ.

A graph of $\lambda(M, \gamma)$, computed by using formula (9), is shown as Fig. 2.

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[1] R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience, 1948.

[2] H. W. Liepmann and A. E. Puckett, *Introduction to aerodynamics of a compressible fluid*, John Wiley and Sons, 1947, 51-60.

[3] J. M. Walsh, R. G. Shreffler and F. J. Willig, *Limiting Conditions for jet formation in high velocity collisions*, J. Appl. Phys. 24 (1953), 349-359.

ON THE THEORY OF THE BULGE TEST[1]

By E. W. ROSS, JR. and W. PRAGER (*Brown University*)

Summary. It is shown that the use of Tresca's yield condition and the associated flow rule leads to a simple theory for the bulge test for perfectly plastic or strain-hardening materials. The basic equations can be integrated in closed form even for finite deflections.

1. Introduction. The ductility of sheet metal under balanced biaxial tension is determined by the bulge test: a circular sheet of uniform thickness is clamped round the periphery and subjected to unilateral fluid pressure which causes the sheet to bulge plastically. The strain at the pole of the bulge is measured by means of a grid inscribed on the originally flat sheet, and the stress at the pole is computed from the applied pressure and the measured curvature and thickness of the deformed sheet. The dimensions of the sheet are chosen so that its flexural stiffness is negligible[2]; on the other hand, the strains cannot be treated as infinitesimal.

The first consistent theory of the bulge test was given by Hill.[3] This theory is based on the yield condition and flow rule of v. Mises.[4] It is assumed that the relevant quantities can be represented as power series in the ratio between the maximum deflection and the radius of the die aperture through which the sheet is made to bulge. Powers higher than

[1]Received Nov. 13, 1953. The results presented in the paper were obtained in the course of research sponsored by Watertown Arsenal Laboratory under Contract DA-19-020-ORD-2598.

[2]Even for a very thin sheet neglecting the flexural stiffness may not be justified in the neighborhood of the edge. Such edge effects are known to be highly localized, however, and may therefore be neglected in the discussion of the states of stress and strain in the neighborhood of the pole of the bulge. For a discussion of plastic edge effects the reader is referred to a paper by F. K. G. Odqvist (Reissner Anniversary Volume, J. W. Edwards, Ann Arbor, Mich., 1949, p. 449).

[3]R. Hill, Phil. Mag. (7) 41, 1133-1142 (1950).

[4]R. v. Mises, Goettinger Nachrichten 1913, 582-592 (1913).

the second are neglected in Hill's analysis which is therefore restricted to moderate deflections.

The present paper contains an alternative theory of the bulge test. This theory is based on Tresca's yield condition[5] and the associated flow rule[6]; its equations can be solved in closed form without the use of special assumptions concerning the magnitude of the deflection.

2. Yield condition and flow rule. The middle surface of the bulged sheet is a surface of revolution, and the principal stresses at a generic point of this surface are directed along the parallel circle, the meridian, and the normal. In this order the principal stresses will be denoted by σ_c, σ_m, and σ_n; the initial thickness of the sheet will be denoted by h_0 and the radius of the die aperture by a. For the usual small values of the ratio h_0/a, the stress σ_n is much smaller than the stresses σ_c and σ_m. The state of stress at any point of the sheet is therefore essentially one of biaxial tension with the principal stresses σ_c and σ_m.

In the following, elastic strains will be neglected and the sheet material will be treated as incompressible. The principal plastic strain rates will be denoted by ϵ_c, ϵ_m, and ϵ_n, and the yield stress in simple tension by σ.

Tresca's yield condition specifies that, for plastic flow to occur, the maximum shearing stress must have an intensity, $\sigma/2$, that depends on the state of hardening of the material. If the three principal stresses are unequal, the state of flow is supposed to be one of pure shear in the plane determined by the largest and smallest principal stresses. For the states of biaxial tension considered here, the following basic possibilities must be considered:

a) If $\sigma_c > \sigma_m > 0$ during plastic flow, then

$$\sigma_c = \sigma \quad \text{and} \quad \epsilon_m = 0, \quad \epsilon_c = -\epsilon_n > 0;$$

b) if $\sigma_m > \sigma_c > 0$ during plastic flow, then

$$\sigma_m = \sigma \quad \text{and} \quad \epsilon_c = 0, \quad \epsilon_m = -\epsilon_n > 0.$$

Finally, it is assumed that any combination of the flow mechanisms specified under a) and b) is possible when $\sigma_c = \sigma_m$. Thus, a third case is added to the list:

c) if $\sigma_c = \sigma_m > 0$ during plastic flow, then

$$\sigma_c = \sigma_m = \sigma \quad \text{and} \quad \epsilon_c > 0, \quad \epsilon_m > 0, \quad \epsilon_n = -(\epsilon_c + \epsilon_m).$$

The flow rule c) is a natural extension of the flow rules a) and b); in fact, it represents the only way in which a continuous transition from a) to b) can be achieved.

For the purpose of comparison, we state the yield condition and flow rule of v. Mises for biaxial tension with the principal stresses σ_c, σ_m and the principal strain rates ϵ_c, ϵ_m:

$$\sigma_c^2 - \sigma_c\sigma_m + \sigma_m^2 = \sigma^2,$$

$$\frac{\epsilon_c}{\epsilon_m} = \frac{2\sigma_c - \sigma_m}{2\sigma_m - \sigma_c}, \qquad \text{sign } \epsilon_c = \text{sign } (2\sigma_c - \sigma_m).$$

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[5]H. Tresca, Mémoires prés. par div. savants, **18**, 733-799 (1868).

[6]W. Prager, *On the use of singular yield conditions and associated flow rules*, J. Appl. Mech. **20**, 317-320 (1953).

This yield condition and flow rule has the great advantage of being valid throughout the entire range of biaxial tension, thus avoiding the necessity of discussing separate cases such as a), b), c) above. On the other hand, each of these three cases has the advantage that the values of two of the quantities σ_c , σ_m , ϵ_c , ϵ_m are known outright. This fact is responsible for the considerable mathematical simplifications obtained by the use of Tresca's yield condition and the associated flow rule which was first pointed out by Koiter.[7]

3. Kinematical considerations. It will be shown in Section 4 that the formation of a spherical bulge of uniform thickness is compatible with Tresca's yield condition and the associated flow rule. In this section, the kinematical relations will be derived which apply during the formation of such a bulge.

Figure 1 shows the circular meridian of the middle surface of the bulged sheet:

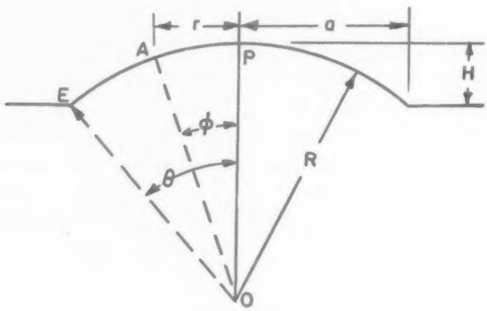

FIG. 1.

O is the center, R the radius, and A a generic point of the meridian; P is the pole, and E represents the edge. The angle POE will be denoted by θ and the angle POA by φ. Since the radius a of the die aperture is given and

$$R = a/\sin \theta, \tag{1}$$

the considered stage of the deformation is conveniently specified by the parameter θ. All "rates" used in the following are rates of change with respect to θ.

Denote by h the uniform sheet thickness in the bulged state and by h_0 that in the undeformed state. Since the sheet material is assumed to be incompressible, it must occupy equal volumina in both states. It follows from this condition that

$$h = h_0 \cos^2 \frac{\theta}{2} . \tag{2}$$

The "strain rate" ϵ_n is thus given by

$$\epsilon_n = \frac{1}{h} \frac{dh}{d\theta} = -\tan \frac{\theta}{2} . \tag{3}$$

[7] W. T. Koiter, C. B. Biezeno Anniversary Volume, H. Stam, Haarlem, Holland, 1953, p. 232.

Since the height of the bulge is

$$H = R(1 - \cos\theta) = a \tan\frac{\theta}{2}, \tag{4}$$

we have

$$\epsilon_n = -H/a. \tag{5}$$

Let r be the radius of the parallel circle through A and r_0 the radius of the corresponding circle in the undeformed sheet. When the condition of incompressibility is applied, not to the entire sheet but only to the portion which is bounded by the circle of radius r_0 in the undeformed state and by the circle of radius r in the deformed state, the following relation is obtained:

$$r = r_0\left(\frac{h_0}{h}\right)^{1/2}, \qquad \cos\frac{\varphi}{2} = r_0\frac{\cos\varphi/2}{\cos\theta/2}. \tag{6}$$

Now,

$$r = R\sin\varphi = a\frac{\sin\varphi}{\sin\theta}. \tag{7}$$

Eliminating r between (6) and (7) and solving for r_0 we obtain

$$r_0 = a\frac{\sin\varphi/2}{\sin\theta/2}. \tag{8}$$

For a given particle r_0 remains constant during the deformation. Differentiating (8) with respect to θ and using (6) and (8), we therefore obtain

$$\frac{d\varphi}{d\theta} = \frac{\tan\varphi/2}{\tan\theta/2} = \frac{r_0^2}{ar}. \tag{9}$$

From (6) and (9) it is seen that the circumferential strain rate is given by

$$\epsilon_c = \frac{1}{r}\frac{dr}{d\theta} = \frac{d}{d\theta}(\log r) = \frac{1}{2}\left[\tan\frac{\theta}{2} - \cot\frac{\theta}{2}\tan^2\frac{\varphi}{2}\right]. \tag{10}$$

From (3), (10), and the condition of incompressibility, the meridional strain rate is then obtained as

$$\epsilon_m = -\epsilon_c - \epsilon_n = \frac{1}{2}\left[\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\tan^2\frac{\varphi}{2}\right]. \tag{11}$$

The circumferential strain rate vanishes at the edge ($\varphi = \pm\theta$); for relevant values of θ, say $0 < \theta < \pi/2$, and all values of φ between $-\theta$ and θ, the strain rates (10) and (11) are seen to be positive. The state of flow considered here therefore is of the type c), and $\sigma_c = \sigma_m = \sigma$.

4. Static considerations. If $\sigma_c = \sigma_m = \sigma$, the equilibrium of the assumed spherical bulge of constant wall thickness under the applied pressure p requires that σ has the constant value

$$\sigma = \frac{pR}{2h} = \frac{pa}{2h_0\sin\theta\cos^2\theta/2} \tag{12}$$

in the entire bulge. This means that, at any given stage of the bulging, the entire material is in the same state of hardening.

Consider first a perfectly plastic material that flows under the constant stress σ_0 in simple tension. We then have $\sigma = \sigma_0 =$ const., and hence, from (12),

$$p = \frac{4\sigma_0 h_0}{a} \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2} . \tag{13}$$

As θ increases, the pressure (13) reaches the maximum value

$$\max p = \frac{3\sqrt{3}\, \sigma_0 h}{4a} \tag{14}$$

when $\theta = 60°$. The considered spherical bulge of constant wall thickness is not stable beyond this pressure maximum. When the pressure maximum is reached, $h/h_0 = 3/4$ according to (2). The logarithmic strain in the direction normal to the sheet therefore is nearly -30%.

Consider next a material that strain-hardens according to

$$\sigma = \sigma_0(1 + \alpha \mid \mathcal{E} \mid) \tag{15}$$

in simple tension, $\mathcal{E}$ being the logarithmic strain, and σ_0 and α being constants. The considered state of stress in the bulged sheet has the principal values $\sigma_c = \sigma_m = \sigma$, $\sigma_n = 0$; it may be obtained by the superposition of the state of balanced triaxial tension $\sigma_c = \sigma_m = \sigma_n = \sigma$ and the state of simple compression $\sigma_c = \sigma_m = 0$, $\sigma_n = -\sigma$. The first of these states of stress will not produce plastic flow or strain-hardening of the incompressible material, the second may be assumed to produce strain hardening in accordance with (15) provided that $\mathcal{E}$ in this formula is replaced by log (h/h_0), where h/h_0 is given by (2). Substituting

$$\sigma = \sigma_0\left(1 - 2\alpha \log \cos \frac{\theta}{2}\right) \tag{16}$$

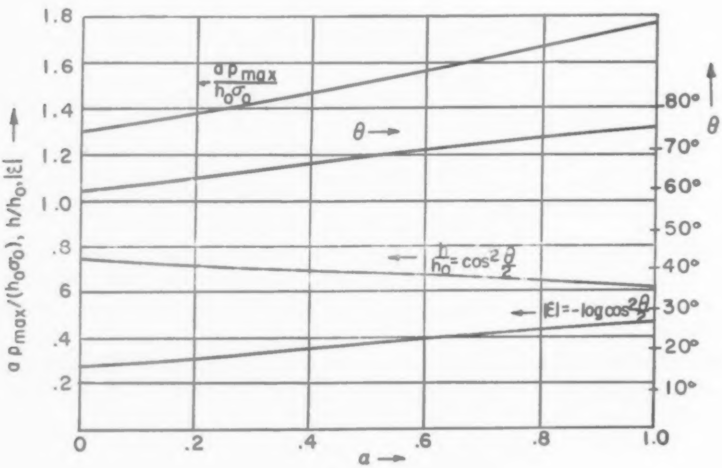

FIG. 2.

into (12) and solving for p, we obtain

$$p = \frac{4\sigma_0 h_0}{a}\left(1 - 2\alpha \log \cos \frac{\theta}{2}\right) \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2} . \tag{17}$$

Figure 2 shows the pressure maximum computed from (17) and the corresponding values of θ, h/h_0, and $| \varepsilon | = -\log (h/h_0)$, all versus the strain-hardening parameter α. It is seen that the pressure maximum p_{max} as well as the corresponding values of θ and $| \xi |$ increase with α, whereas the ratio h/h_0 at the pressure maximum decreases with α. In the considered range of α all these quantities vary with α in a nearly linear manner.

BOOK REVIEWS

Mathematical methods for scientists and engineers. By Lloyd P. Smith. Prentice-Hall, Inc., New York, 1953. x + 453 pp. $10.00.

The material in this book is essentially that taught for a number of years by the author to graduate students in physics and physical chemistry at Cornell University. The unusual feature of this text is the wide range of mathematical methods treated. At the same time it will be found that the treatment of the material is quite adequate for dealing with most physical problems. The discussion is consise and clear.

There is no specific treatment of differential equations except as they arise in the treatment of the other topics.

The reviewer believes that this is one of the most useful books available on intermediate and advanced mathematical methods.

The chapter headings of this text are as follows: Elements of Function Theory; Differential Calculus, Integral Calculus; Space Geometry; Line, Surface, and Multiple Integrals; Theory of Functions of a Complex Variable Residues and Complex Integration; Representation of Functions by Infinite Series of Functions; Applications of Functions of a Complex Variable to Potential and Flow Problems; Algebra of Linear Equations, Transformations and Quadratic Forms; Vector and Tensor Analysis; Orthonormal Function Systems; Orthonormal Functions with a Continuous Spectrum; Integral Equations; Variational Methods; and Elements of Probability Theory.

ROHN TRUELL

Calculus of variations with applications to physics and engineering. By Robert Weinstock. McGraw-Hill Book Company, Inc., New York, 1952. x + 326 pp. $6.50.

According to the preface, this volume presents an introduction to the calculus of variations followed by application of the subject to problems of physics and theoretical engineering.

The first five chapters give the usual elementary treatment of the calculus of variations with no pretense of complete mathematical rigor. Chapters 6-12 present applications to dynamics, elasticity, quantum mechanics, and electrostatics. These applications are, for the greater part, of an elementary nature; modern problems in acoustics, electromagnetic theory, and quantum mechanics are not discussed.

Up to the present time there is no other volume in the English language that offers such a variety of applications of the calculus of variations to problems in physics. The book must therefore be accepted as a worthwhile contribution to the applied mathematician's library.

This reviewer has a few adverse comments to make on the contents of the book; these are not in-

tended to detract from the general acceptability of this volume as a text-book at the first-year graduate level.

(a) It is claimed in the preface that "the reader who has mastered the essence of the material included should have little difficulty in applying the calculus of variations to most of the subjects which have been squeezed out." This is unwarranted optimism on the part of Professor Weinstock. The book does not offer any examples of variational principles for integral equations; there is no mention of Green's functions. How, then, could a reader be expected to understand "with little difficulty" the Levine-Schwinger treatment of scattering problems?

(b) There is no mention of variational techniques for obtaining lower bounds to eigenvalues (work of Weinstein, Aronszajn, etc.). In particular the chapter on quantum mechanics is weak because of this and other omissions.

(c) The work of Pólya and Szegö and their colleagues on "isoperimetric inequalities in mathematical physics" is barely touched on in the last chapter on electrostatics.

(d) Chapter 5 is entitled "Geometrical Optics: Fermat's Principle." This chapter is five pages long and consists exclusively of deriving the two-dimensional principle of Fermat by two different methods. It is unfortunate that the author did not discuss "corner conditions" in this connection. Snell's law could then be taken as an illustration and the chapter expanded to reasonable size. (See e.g. H. Lewy: "Aspects of the Calculus of Variations." University of California Press, Berkeley, California, 1939. This excellent little volume is not listed in Weinstock's bibliography.)

(e) A few serious errors in the text have been pointed out by Synge in his review in "Bulletin of the American Mathematical Society", Vol. 59, No. 4, July, 1953.

Here are two additional non-trivial errors.

1. The second part of theorem (d), p. 157, is incorrect.

2. Introduction, p. 2: "The only values of x at which $y = g(x)$ can possibly achieve a minimum are the roots of $g'(x) = 0$." A minimum can occur at points where $g'(x)$ does not exist, or at an endpoint of an interval. This and similar precautions are neglected throughout the book; for instance, on p. 39, where no condition on the differentiability of $g(x, y)$ is given.

I. Stakgold

Numerische Behandlung von Differentialgleichungen. By L. Collatz. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1951. xiii + 458 pp. $11.43.

This book represents a careful and comprehensive study of the numerical methods of solution of ordinary and partial differential equations. It also contains a concise treatment of numerical methods for integral equations. By virtue of the wealth of material covered the book should prove to be a very valuable source of reference to anyone interested in this important subject.

In addition to a description of the various methods the author has included, particularly in the case of ordinary differential equations, analyses of their theoretical foundations. In many instances there is a discussion of the estimation of the error. There are numerous illustrative examples, many of them representing practical physical problems.

Chapter I contains, in addition to introductory remarks, difference formulae and quadrature formulae, a study of initial-value problems for ordinary equations of first and higher order. Chapter II is devoted to boundary-value problems (boundary conditions prescribed at more than one point) for ordinary equations, including eigenvalue problems. In Chapter III we find initial-value problems and combined initial and boundary-value problems for partial differential equations. In chapter IV the author takes up boundary-value problems for partial differential equations. Chapter V is concerned with integral equations and a brief discussion of functional equations.

G. W. Morgan

Die praktische Behandlung von Integralgleichungen. By H. Bückner. Springer-Verlag, Berlin-Goettingen-Heidelberg, 1952. vi + 127 pp. DM 18.60.

As the author states in the Introduction, the numerical treatment of integral equations is a comparatively new and still developing field of numerical analysis. The present systematic survey is therefore particularly welcome even though it is restricted to equations of the Fredholm type for functions of a single variable. The first chapter recapitulates the principal results of the general theory. Chapter II presents formulas and variational principles that are useful in the calculation of characteristic values. Particular attention is given to methods of bracketing characteristic values. Chapter III is devoted to iterative methods. The use of mixed iteration for the solution of inhomogeneous integral equations is of special interest. Chapter IV is concerned with techniques using approximations to the kernel of the integral equation. Perturbation methods, the methods of Ritz and Galerkin, and methods replacing the integral by a sum are treated as special cases. The brief final chapter deals with special kernels.

W. PRAGER

Mathematical physics. By Donald H. Menzel. Prentice-Hall, Inc., New York, 1953 v + 412 pp. $8.50.

This text contains five chapters entitled: I. Physical Dimensions and Fundamental Units, II. Mechanics and Dynamics, III. Waves and Vibrations, IV. Classical Electromagnetic Theory, V. Relativity. The book is claimed by the author to be designed for use in junior, senior, or graduate courses in mathematical physics; this purpose seems to have been achieved very nicely. The book appears to the reviewer to be a very satisfactory text for the transition from advanced undergraduate physics to first year graduate physics, and it should be welcomed by many students and teachers.

This is not a text on mathematical methods in physics. It is rather a text on theoretical physics with emphasis on the mathematics.

ROHN TRUELL

The principles of the control and stability of aircraft. By W. J. Duncan. Cambridge University Press, 1952. xvi + 384 pp. $8.00.

In view of the importance of the field and the abundance of texts on other aspects of aeronautics, the scarcity of textbooks on aircraft stability and control is surprising. The present volume is therefore a particularly welcome addition to the aeronautical literature. Within the limited space available for this review, the scope of the work is best indicated by the following chapter headings: Introductory survey. Elementary mechanics of flight. The equations of motion of rigid aircraft. Methods for solving the dynamical equations and for investigating stability. Longitudinal-symmetric motion. Lateral-antisymmetric motion. Flap controls in general. The measurement of aerodynamic derivatives. Controls for roll, pitch, and yaw. Static stability and maneuverability. Stalling and the spin (by A. D. Young). The influence of distortion of the structure. The influence of the compressibility of the air. Flaps for landing and take-off (by A. D. Young). Sundry topics.

The exposition is particularly clear and easy to follow.

W. PRAGER

CORRECTION

In the review of Petrovskij, *Vorlesungen ueber die Theorie Integralgleichungen,* this Quarterly **11**, 375 (1953) the price of this book was erroneously stated to be $7.80 instead of DM 7.80.

Complex analysis. An introduction to the theory of analytic functions of one complex variable. By Lars V. Ahlfors. McGraw-Hill Book Co., New York, Toronto, London, 1953. xi + 247 pp. $5.00.

This book is a rigorous introductory treatment of the theory of analytic functions of one complex variable. It is very readable and would be an excellent basis on which to teach a course. There are exercises every few pages. Although no physical examples are given, the student who has mastered this book will have no mathematical difficulty applying the knowledge he has gained to any application of complex variable in mathematical physics.

In Chapter 1, complex numbers are introduced, their geometrical representation and linear transformations considered. Chapter 2 is devoted in the main to theorems needed from the theory of a real variable and from the theory of sets. Complex integration is introduced in Chapter 3. Cauchy's integral formula and the calculus of residues follow. Chapter 4 contains convergence and uniform convergence of series, Taylor and Laurent series, infinite products and normal families. The Dirichlet problem is considered in Chapter 5 and multiple-valued functions in Chapter 6.

D. R. Bland

Proceedings of Symposia in Applied Mathematics—Fluid Dynamics. Volume IV. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1953, for the American Mathematical Society, 80 Waterman Street, Providence, Rhode Island. v + 186 pp. $7.00.

This volume contains thirteen papers and one abstract of a paper which were presented at the Fourth Symposium in Applied Mathematics of the American Mathematical Society held at the University of Maryland on June 22 and 23, 1951. All papers are concerned with fluid dynamics. Topics in turbulence, compressible (including transonic), potential, and viscous flow are treated, and one paper deals with the development of the hydrodynamical equations from the point of view of thermodynamics of irreversible processes.

The papers included are: S. Chandresekhar, "Some aspects of the statistical theory of turbulence". C. C. Lin, "A critical discussion of similarity concepts in isotropic turbulence". A. Busemann, "The nonexistence of transonic potential flow". R. E. Meyer, "On waves of finite amplitude in ducts (abstract)". T. Y. Thomas, "On the problem of separation of supersonic flow from curved profiles". G. F. Carrier and K. T. Yen, "On the construction of high-speed flows". M. H. Martin and W. R. Thickstun, "An example of transonic flow for the Tricomi gas". A. E. Heins, "On gravity waves". S. R. De Groot, "Hydrodynamics and Thermodynamics". J. M. Burgers, "Non-uniform propagation of plane shock waves". T. Theodorsen, "Theory of propellers". G. Birkhoff, D. M. Young and E. H. Zarantonello, "Numerical methods in conformal mapping". J. L. Synge, "Flow of viscous liquid through pipes and channels". A. Weinstein, "The method of singularities in the physical and in the hodograph plane".

George W. Morgan

Anfangswertprobleme bei partiellen Differentialgleichungen. By R. Sauer. Springer-Verlag, Berlin-Goettingen-Heidelberg, 1952. xiv + 229 pp. $6.90.

The book treats initial value problems for single partial differential equations and systems of partial differential equations of hyperbolic type. The introductory first chapter discusses the typical boundary value problem for the Laplace equation and the typical initial value problem for the wave equation. The second chapter develops the theory of characteristics for first partial differential equations of the first order. Chapters III and IV which constitute about three quarters of the volume are devoted to the system of quasilinear differential equations of the first order and the equation of the second order, Chapter III treating the case of two and Chapter IV the case of more than two independent variables.

The clear exposition makes the book easy to read. A particularly valuable feature is the constant detailed reference to relevant physical problems (hydrodynamics, acoustics, gas dynamics, plasticity).

W. Prager

SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written [illegible] professional titles and degrees or honors [illegible]. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ('), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set above letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent ½ rather than with the sign $\sqrt{\ }$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp[(a^2+b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

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